

RIGHT-ANGLED BILLIARDS AND VOLUMES OF MODULI SPACES OF QUADRATIC DIFFERENTIALS ON \mathbb{CP}^1

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ABSTRACT. We use the relation between the volumes of the strata of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 and counting functions of the number of (bands of) closed geodesics in associated flat metrics with singularities to prove a very explicit formula for the volume of each such stratum conjectured by M. Kontsevich a decade ago.

Applying ergodic techniques to the Teichmüller geodesic flow we obtain weak quadratic asymptotics for the number of (bands of) closed trajectories and for the number of generalized diagonals in almost all right-angled billiards.

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1. INTRODUCTION

1.1. Counting trajectories of right-angled billiards. Currently it is not known whether there exists a single closed billiard trajectory in every obtuse triangle (see [S08] for some progress in this direction and for further references). The situation with billiards in *rational* polygons (that is in polygons with angles which are rational multiples of π) is understood much better: trajectories of such billiards are related to geometry of certain compact flat surfaces with conical singularities, which are thoroughly studied starting with the landmark papers of H. Masur [M82] and W. Veech [Ve82]. In particular, it is known that a billiard in any rational polygon has infinitely many closed trajectories [KMS86], and furthermore the number of trajectories of length at most L is bounded between $c_1 L^2$ and $c_2 L^2$ for some $0 < c_1 < c_2$ and for L large enough, see [M88] and [M90].

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In the current paper we study families of right-angled billiards like the ones in Figure 1. Namely, we assume that the billiard table is a topological disk endowed with a flat metric, and that the boundary of the disk is piecewise geodesic such that the angle at every corner of the boundary is an integer multiple of $\frac{\pi}{2}$. We consider families of polygons sharing the same interior corner angles $(\frac{\pi}{2}k_1, \frac{\pi}{2}k_2, \dots, \frac{\pi}{2}k_n)$. Actually, it will be convenient to consider a slightly larger space $\mathcal{B}(k_1, \dots, k_n)$ of “directional billiards” distinguishing a billiard table Π and the same table turned by angle ϕ . The measure in the space $\mathcal{B}(k_1, \dots, k_n)$ is the product measure of Lebesgue measure arising from the side lengths and the angular measure $d\phi$.

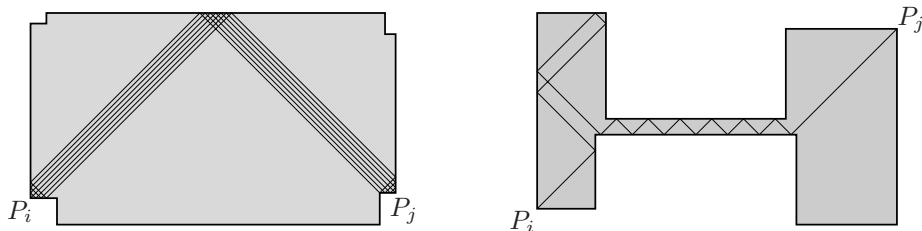


FIGURE 1. A family $\mathcal{B}(k_1, \dots, k_n)$ of right-angled polygons; a band of periodic trajectories on the left, and a generalized diagonal on the right.

We count the number of generalized diagonals of bounded length in such billiards (that is, the number of trajectories of bounded length which start in some fixed corner P_i and arrive to some fixed corner P_j , see Figure 1) and the number of closed billiard trajectories of bounded length. Note, that closed regular trajectories are never isolated in rational billiards: they always form bands of “parallel” closed trajectories of the same length, see Figure 1. Thus, when counting closed trajectories one actually counts the number of such bands. Sometimes, it is natural to count the bands with a weight which registers the thickness of the band, see e.g. Theorem 1.4 at the end of §1.1. By convention we always count *non-oriented* generalized diagonals and *non-oriented* closed billiard trajectories.

To give an idea of the general theorems stated in detail in §2 and developed in §4, we present the following representative results. (The precise meaning of the “weak asymptotics” in the statement below is defined in §2.2).

Theorem 1.1. *For any right-angled billiard Π outside of a zero measure set in any family $\mathcal{B}(k_1, \dots, k_n)$ the number $N_{ij}(\Pi, L)$ of generalized diagonals of length at most L joining a pair of fixed corners P_i, P_j with angles $\frac{\pi}{2}$ has the following weak quadratic asymptotics as $L \rightarrow \infty$:*

$$(1.1) \quad N_{ij}(\Pi, L) \sim \frac{1}{2\pi} \cdot \frac{L^2}{\text{Area of the billiard table}}.$$

Theorem 1.1 is proved in §4.11.

The fact that this asymptotics does not depend at all on the billiard table is at the first glance counterintuitive. What is even more surprising is that it is a universal constant: it is the same not only for almost all billiard tables inside each family, but it does not vary even from one family to another! In particular, though the shape of the two polygons of the same area in Figure 1 is quite different, the number of trajectories of length at most L joining the right-angle corner P_i to the

right-angle corner P_j is approximately the same in both cases, and is approximately the same as the number of trajectories of length at most L joining two corners of the usual rectangular billiard of the same area when $L \gg 1$.

The situation becomes more complicated when we consider other types of corners of the billiard. Consider, for example, an L-shaped billiard table as on Figure 2. Let P_1, \dots, P_5 be the right-angle corners of the L-shaped billiard, and let P_0 be the

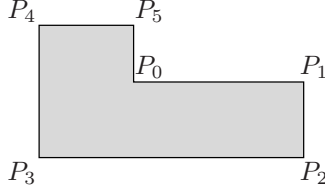


FIGURE 2. L-shaped billiard.

corner with the interior angle $\frac{3\pi}{2}$.

Theorem 1.2. *For almost any L-shaped billiard Π the number $N_{i0}(\Pi, L)$ of generalized diagonals of length at most L joining a fixed corner P_i with angle $\frac{\pi}{2}$ and the corner P_0 with angle $\frac{3\pi}{2}$ has the following weak quadratic asymptotics as $L \rightarrow \infty$:*

$$(1.2) \quad N_{i0}(\Pi, L) \sim \frac{2}{\pi} \cdot \frac{L^2}{\text{Area of the billiard table}}.$$

The naive intuition does not help: the angle $\frac{3\pi}{2}$ at the corner P_0 is *three* times larger than in the previous case, while the constant in the asymptotics for the number of generalized diagonals is *four* times larger than in the previous statement. Currently we have no idea how to obtain this factor 4 without using techniques of the Teichmüller geodesic flow, Lyapunov exponents of the Hodge bundle, and the computation of volumes of the moduli spaces of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 . Theorem 1.2 is proved in §4.9.

Using recently developed technology, one can prove weak asymptotic formulas like Theorem 1.1 and Theorem 1.2 for individual billiard tables. In particular, the following holds:

Theorem 1.3. *Suppose Π is an L-shaped billiard table as in Figure 2. Let*

$$a = \frac{|P_3P_4|}{|P_1P_2|}, \quad b = \frac{|P_2P_3|}{|P_4P_5|}.$$

Then,

- (i) *If a and b are both rational, or if there exists a non-square integer $D > 0$ such that $a, b \in \mathbb{Q}(\sqrt{D})$ and $a + \bar{b} = 1$ (where \bar{b} is the Galois conjugate of b), then*

$$(1.3) \quad N_{ij}(\Pi, L) \sim c_{ij} \frac{L^2}{\text{Area of the billiard table}},$$

- (ii) *For any other L-shaped billiard table, (1.1) and (1.2) hold.*

In the case (i) the Siegel–Veech constants c_{ij} for rational values of parameters a, b can be computed by the formula due to E. Gutkin and C. Judge [GJ]. For $i, j \neq 0$ and $a, b \in \mathbb{Q}(\sqrt{D})$ the constants c_{ij} are computed by M. Bainbridge, see [Ba07, Theorem 1.5 and §14].

Proof. Theorem 1.3 is a compilation of several different results. In case (i), the polygon Π is a Veech polygon, which gives rise to a Teichmüller curve, see [C04], [Mc03]. The existence of an asymptotic formula such as (1.3) for such a situation was proved in the pioneering work of W. Veech [Ve89].

Let

$$U = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \subset \mathrm{SL}(2, \mathbb{R}), \quad P = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \subset \mathrm{SL}(2, \mathbb{R}).$$

The fact that weak asymptotic formulas such as those of part (ii) hold for any rational billiard table follows from [EMiMo, Theorem 2.12], which uses the general invariant measure classification theorem of [EMi] for the action of P on moduli space. However, to evaluate the constant for an arbitrary L-shaped table, one also has to appeal to the explicit classification of $\mathrm{SL}(2, \mathbb{R})$ -invariant affine submanifolds in the moduli space of Abelian differentials in genus 2 due to C. McMullen, [Mc07]. \square

We note that asymptotic counting formulas for individual billiards are associated with invariant measure classification theorems on the action of subgroups of $\mathrm{SL}(2, \mathbb{R})$ on (certain subsets of) the moduli space. In particular, when a measure classification theorem for the action of the subgroup U exists (e.g. in the case of a Teichmüller curve), one can get a strong asymptotic formula. Also, a measure classification theorem for the action of the subgroup P leads to a weak asymptotic formula. For other examples when a classification of invariant measures for the action of U (and thus strong asymptotic formulas) are known see [EMS03], [EMM06], [CW10], [Ba10].

For the best currently known counting statements for other types of right-angled billiard tables, see §2.2. In this paper, our focus is on the evaluation of the constants rather than on the proof of various forms of asymptotics for individual billiards. However, in §3 and §6 we give fairly self-contained proofs of weak asymptotics for almost all right-angled billiards in a family such as Theorem 1.1 and Theorem 1.2. See the end of §1.2 for a discussion of this argument.

We complete this section with an illustration of further counting problems where one can apply our techniques. Let $N_{area}(\Pi, L)$ denote the number of bands of closed periodic billiard trajectories of length at most L counted with a weight given by the normalized area of the band. More precisely, we count the area of overlapping domains of the band twice: the area of the band is naively measured as the area of the associated cylinder on the flat sphere, that is, the width of the band times the length of the closed trajectory, normalized by the area of the billiard table. Having measured the area of the band, we divide it by the area of the billiard table to get the weight of the band.

Theorem 1.4. *For almost any billiard Π in any family $\mathcal{B}(k_1, \dots, k_n)$ of right-angled billiards the weighted number of bands of closed billiard trajectories of length at most L satisfy the following weak asymptotics as $L \rightarrow \infty$:*

$$(1.4) \quad N_{area}(\Pi, L) \sim \frac{1}{16\pi} \sum_{j=1}^n \left(\frac{4}{k_j} - k_j \right) \cdot \frac{L^2}{\text{Area of the billiard table}}.$$

Theorem 1.4 is proved in §6.1. The constant in the corresponding counting function is directly related to the *Siegel–Veech area constant* for the corresponding stratum of meromorphic quadratic differentials on \mathbb{CP}^1 discussed in §1.3.

1.2. Right-Angled billiard tables and quadratic differentials. Given a right-angled billiard Π in $\mathcal{B}(k_1, \dots, k_n)$ we can glue a topological sphere from two superposed copies of Π identifying the boundaries of the two copies by isometries, see Figure 3. By construction the resulting topological sphere is endowed with a flat metric. Note that the metric is regular on interior of the segments coming from the boundary of Π : one can unfold a neighborhood of any such point into a small regular flat domain.

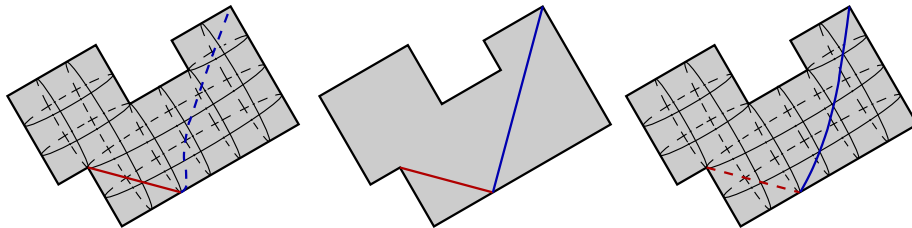


FIGURE 3. Flat spheres glued from two copies of a right-angled billiard. The angle by which the billiard table is rotated with respect to the horizontal position encodes the “phase” of the corresponding quadratic differential. A general generalized diagonal in the polygon gives rise to two distinct saddle connections on the flat sphere.

However, the resulting flat metric has conical singularities with cone angles $\pi k_1, \dots, \pi k_n$ at the points coming from the vertices of Π . By construction the linear holonomy of the flat metric with isolated singularities belongs to the group $\mathbb{Z}/2\mathbb{Z}$: the parallel transport along a short path encircling a conical point P_j brings a tangent vector \vec{v} either to itself or to $-\vec{v}$ depending on the parity of k_j .

It is known that a flat metric with isolated conical singularities and with holonomy in $\mathbb{Z}/2\mathbb{Z}$ on a closed surface defines a complex structure and a meromorphic quadratic differential q in this complex structure defined up to multiplication by a scalar $e^{i\phi}$. Choosing a line direction $\pm\vec{v}$ at some point of the resulting flat sphere as a “horizontal” direction we fix the scalar $e^{i\phi}$. In an appropriate flat local coordinate z outside of the conical points the resulting quadratic differential has the form $(dz)^2$. A conical singularity with a cone angle $k_i\pi$ corresponds to a zero of the quadratic differential of degree $k_i - 2$, where a “zero of degree -1 ” is a simple pole.

Actually, the two structures are synonymous: a meromorphic quadratic differential q with at most simple poles on a Riemann surface defines a canonical flat metric with isolated conical singularities, with linear monodromy in $\mathbb{Z}/2\mathbb{Z}$ and with a distinguished foliation by straight lines in the flat metric (see the original papers [M82] and [Ve82] or surveys [MT99] and [Z03]).

By construction closed billiard trajectories in Π are in canonical one-to-two correspondence with closed regular geodesics on the associated flat sphere, and generalized diagonals on Π are in the natural one-to-two correspondence with the *saddle connections* on the associated flat sphere, see Figure 3. Thus, the two counting problems are closely related.

It is known by work of Veech [Ve98] and of Eskin-Masur [EM00] that almost all flat spheres in a given stratum $\mathcal{Q}(d_1, \dots, d_n)$ satisfy a quadratic asymptotic

formula for the number of saddle connections. However, we cannot immediately translate this result to right-angled billiards. An elementary count shows that the space $\mathcal{B}(k_1, \dots, k_n)$ has *real* dimension $n - 2$, while the associated stratum $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$ has *complex* dimension $n - 2$. Thus, flat spheres constructed from right-angled billiards form a subset of measure zero, and “almost all” results for the strata are not applicable to families of billiards. This is the common difficulty of translating results valid for flat surfaces to billiards.

In our specific case we are lucky enough to get a subspace of flat spheres “of billiard origin” which is transversal to the unstable foliation of the Teichmüller flow (see §3). This allows us to apply certain standard techniques of hyperbolic dynamics to obtain some ergodic results in slightly weaker form. As a corollary we obtain the desired information on “weak quadratic asymptotics” in the counting problems for almost all billiards. The corresponding ergodic technique is presented in §6.

1.3. Volumes of moduli spaces of quadratic differentials. We start by solving the counting problems for quadratic differentials. The Siegel–Veech constant c_{area} responsible for the exact quadratic asymptotics of the weighted number of bands of regular closed geodesics on almost any flat sphere in a given stratum $\mathcal{Q}(d_1, \dots, d_n)$ of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 was recently computed in [EKZ],

$$(1.5) \quad c_{area}(\mathcal{Q}(d_1, \dots, d_n)) = -\frac{1}{8\pi^2} \sum_{j=1}^n \frac{d_j(d_j + 4)}{d_j + 2}.$$

Developing techniques elaborated in [EMZ03] for the strata of Abelian differentials and using the further results from [Bo] and [MZ08] on the *principal boundary* of the strata of quadratic differentials we express the Siegel–Veech constant c_{area} in terms of the ratio of the volumes of appropriate strata,

$$(1.6) \quad c_{area}(\mathcal{Q}(d_1, \dots, d_n)) = \frac{\text{Explicit polynomial in volumes of simpler strata}}{\text{Vol}(\mathcal{Q}(d_1, \dots, d_n))}.$$

In this way we obtain a series of identities on the volumes of the strata of meromorphic quadratic differentials with at most simple poles in genus zero. The resulting identities recursively determine the volumes of all strata. This allows us to find the volumes of all strata of meromorphic quadratic differentials with at most simple poles in genus 0.

Define the function on integers n greater than or equal to -1 as

$$(1.7) \quad v(n) := \frac{n!!}{(n+1)!!} \cdot \pi^n \cdot \begin{cases} \pi & \text{when } n \text{ is odd} \\ 2 & \text{when } n \text{ is even} \end{cases}$$

for $n = -1, 0, 1, 2, 3, \dots$. Here the double factorial $n!! = n \cdot (n-2) \cdot \dots$ is the product of all even (respectively odd) positive integers smaller than or equal to n . By convention we set

$$(-1)!! = 0!! = 1,$$

which implies that

$$v(-1) = 1 \quad \text{and} \quad v(0) = 2.$$

Theorem 1.5. *The volume of any stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 (i.e. $d_i \in \{-1; 0\} \cup \mathbb{N}$ for $i = 1, \dots, k$, and $\sum_{i=1}^k d_i = -4$) is equal to*

$$(1.8) \quad \text{Vol } \mathcal{Q}_1(d_1, \dots, d_k) = 2\pi^2 \cdot \prod_{i=1}^k v(d_i)$$

(where all the zeroes and poles are “named”).

This formula for the volume (up to some normalization factor) was conjectured by M. Kontsevich about ten years ago. It is much simpler than the formula for the volumes of the strata of Abelian differentials found by A. Eskin and A. Okounkov [EO01]. For the case of strata of quadratic differentials in higher genus no explicit formula for the volumes is currently known (even though some calculations are possible using [EO03]).

The proof of Theorem 1.5, given in §5, consists in verifying that the expression (1.8) for the volume satisfies the combinatorial identities implied by (1.5) and (1.6). Part of this verification is performed in Appendix A.

Note that the convention that all zeroes and poles are “named” affects the normalization: we compute the volumes of the corresponding covers over strata with “anonymous” singularities. For example, the stratum $\mathcal{Q}(1, -1^5)$ of quadratic differentials with “anonymous” zeroes and poles is isomorphic to the stratum $\mathcal{H}(2)$ of holomorphic Abelian differentials; by convention the volume elements are chosen to be invariant under this isomorphism. However, by (1.8) we have

$$\text{Vol } \mathcal{Q}_1(1, -1^5) = 2\pi^2 \cdot v(1) \cdot (v(-1))^5 = 2\pi^2 \cdot \frac{\pi^2}{2} \cdot 1^5 = 5! \cdot \frac{\pi^4}{120} = 5! \cdot \text{Vol } \mathcal{H}_1(2),$$

which corresponds to $5!$ ways to *give names* to five simple poles.

Similarly,

$$\text{Vol } \mathcal{Q}_1(2, -1^6) = 2\pi^2 \cdot v(2) \cdot (v(-1))^6 = 2\pi^2 \cdot \frac{4\pi^2}{3} \cdot 1^6 = \frac{6!}{2!} \cdot \frac{\pi^4}{135} = \frac{6!}{2!} \cdot \text{Vol } \mathcal{H}_1(1, 1).$$

This time there is an extra factor $\frac{1}{2!}$ responsible for *forgetting the names* of the two zeroes of $\mathcal{H}(1, 1)$.

1.4. Counting pillowcase covers. One of the ways to compute the volumes of the strata of Abelian or quadratic differentials (actually, the only one before the current paper) is to count *square-tiled surfaces* or *pillowcase covers*, see [EO01], [EO03], [EOP], [Z00]. In the current paper we follow an alternative method, and, thus, our result implies an explicit expression for the leading term of the function counting associated pillowcase covers, when the degree of the cover tends to infinity.

Namely, following [EO03] we define a *pillowcase cover* of degree $4d$ as a ramified cover

$$\pi : \hat{\mathcal{P}} \rightarrow \mathcal{P}$$

over the pillowcase orbifold $\mathcal{P} = (\mathbb{C}/\mathbb{Z} \oplus i\mathbb{Z})/\pm$ (as in Figure 4) with ramification data given as follows. Let η be a partition and ν a partition of an even number into *odd* parts. Viewed as a map to the sphere, π has profile $(\nu, 2^{2d-|\nu|/2})$ over $0 \in \mathcal{P}$ and profile (2^{2d}) over the other three corners of \mathcal{P} . Additionally, π has

profile $(\eta_i, 1^{4d-\eta_i})$ over $\ell(\eta)$ given points of \mathcal{P} and unramified elsewhere, where $\ell(\eta)$ is the number of parts in η . This ramification data determines the genus g of $\hat{\mathcal{P}}$ by

$$2 - 2g = \chi(\hat{\mathcal{P}}) = \ell(\eta) + \ell(\nu) - |\eta| - |\nu|/2.$$

We consider only those ramification data for which $g = g(\hat{\mathcal{P}})$ in the above formula is equal to zero,

$$(1.9) \quad \ell(\eta) + \ell(\nu) - |\eta| - |\nu|/2 = 2.$$

Denote by $\text{Cov}_{4d}^0(\eta, \nu)$ the number of inequivalent degree $4d$ connected covers

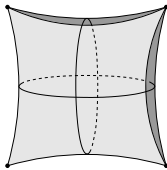


FIGURE 4. Pillowcase orbifold.

$\hat{\mathcal{P}} \rightarrow \mathcal{P}$ with ramification data (η, ν) . Denote by $\mathcal{Q}(\eta, \nu)$ the moduli space of quadratic differentials with singularity data $\{\nu_i - 2\}$ and $\{2\eta_i - 2\}$. Condition (1.9) guarantees that $\mathcal{Q}(\eta, \nu)$ is nonempty, and corresponds to genus zero.

Theorem 1.6. *For any ramification data (η, ν) satisfying condition (1.9) the number $\text{Cov}_{4d}^0(\eta, \nu)$ of pillowcase covers of type (η, ν) admits the following limit:*

$$(1.10) \quad \lim_{N \rightarrow \infty} \frac{1}{N^{\ell(\eta) + \ell(\nu) - 2}} \sum_{d=1}^N \text{Cov}_{4d}^0(\eta, \nu) = \frac{\text{Vol } \mathcal{Q}_1(\eta, \nu)}{2(\ell(\eta) + \ell(\nu) - 2)},$$

where $\text{Vol } \mathcal{Q}_1(\eta, \nu)$ is given by equation (1.8).

Theorem 1.6 is proved in §B.2. We remark that the leading term in the asymptotics for number of pillowcase covers is the same as if we consider the pillowcase covers where all preimages of three corners of the pillow have ramification index 2, and all remaining ramification points project to the fourth corner of the pillow, see [EO01].

Note that the more natural direct geometric approach to the counting of pillowcase covers leads to rather involved combinatorial problems. We present this alternative geometric approach in a separate paper [AEZ].

1.5. Reader's guide. The paper (like Caesar's Gaul) is composed of three parts. The reader interested only in the billiards may read only §2 (and optionally §3 and §6).

The part where we compute the volume of any stratum $\mathcal{Q}_1(d_1, \dots, d_n)$ of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 and where we compute the *Siegel–Veech constants* for these strata is independent from the rest of the paper. It is presented in §2.1, §§3.1–3.2 and in §§4–5 (with one verification in Appendix A).

Finally, Appendix B devoted to pillowcase covers is completely independent of the rest of the paper.

Acknowledgments. The authors are grateful to M. Kontsevich for extremely stimulating discussions. In particular, the formula for the volume of the strata of quadratic differentials was conjectured by M. Kontsevich about a decade ago.

Part of this paper strongly relies on techniques developed in collaboration with H. Masur. We are grateful to him for his very important contribution.

We thank C. Boissy for the list of configurations of homologous saddle connections in genus 0, elaborated specifically for our needs, and for his pictures of configurations.

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2. CONFIGURATIONS AND COUNTING THEOREMS

2.1. Types of saddle connections and generalized diagonals. We distinguish the following four ways of getting generalized diagonals in a right-angled billiard. They correspond to four types of configurations of saddle connections on a flat sphere defined by a meromorphic quadratic differential with simple poles, see [EMZ03] and [MZ08] for general information on *configurations of saddle connections* and [Bo] for specific case of \mathbb{CP}^1 .

I. Saddle connection joining distinct singularities. In this situation (see Figure 5) we have a generalized diagonal joining a corner P_i with the inner angle $k_i \frac{\pi}{2}$, where $k_i \geq 3$, to a distinct corner P_j .



FIGURE 5. Type I. On the left: a generalized diagonal joining two distinct corners of the billiard, where at least one of the two corners has inner angle at least $\frac{3\pi}{2}$. It does not bound a band of closed trajectories. On the right: a saddle connection on \mathbb{CP}^1 joining a zero to a distinct zero (or to a pole).

The induced flat metric on \mathbb{CP}^1 has an associated saddle connection of the same length joining the zero P_i to the distinct zero (or simple pole) P_j .

II. Saddle connection joining a zero to itself. This situation (see Figure 6) can happen only when we have a corner P_i with a corner angle $k_i \frac{\pi}{2}$ with $k_i \geq 4$. In this case we can have a generalized diagonal joining the corner P_i to itself such that it does not bound a band of closed regular trajectories.

For the induced flat metric on \mathbb{CP}^1 we get a corresponding saddle connection of the same length joining the zero P_i to itself such that the total angle $k_i \pi$ at the singularity P_i is split by the separatrix loop into two sectors having the angles strictly greater than π (which is equivalent to the condition that generically such a saddle connection does not bound a cylinder filled with periodic geodesics).



FIGURE 6. Type II. On the left: a generalized diagonal returning to the same corner. For this type, it does not bound closed trajectories. On the right: the corresponding saddle connection joining a zero (of order at least 2) to itself.

III. A “pocket”. In this situation (see Figure 7) we have a band of periodic trajectories. The boundary of the band is composed of two generalized diagonals. The first generalized diagonal joins a pair of corners P_i, P_j with inner angles $\frac{\pi}{2}$. The length of this saddle connection is twice shorter than the length of periodic billiard trajectory in the band. The second generalized diagonal joins a corner P_i with inner angle $k_i \frac{\pi}{2}$ with $k_i \geq 3$ to itself. The length of this saddle connection is the same as the length of periodic billiard trajectory in the band.

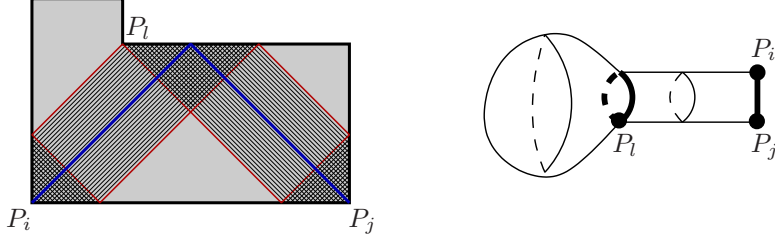


FIGURE 7. Type III. On the left: a band of closed trajectories bounded by two generalized diagonals. One of generalized diagonals joins two distinct corners with angles $\frac{\pi}{2}$; the other returns to the same corner. On the right: the corresponding “pocket” configuration with a cylinder bounded on one side by a saddle connection joining two simple poles, and by a saddle connection joining a zero to itself on the other side.

For the associated flat metric on \mathbb{CP}^1 we get a cylinder filled with closed regular trajectories. One of the boundary components of the cylinder degenerates to a saddle connection joining two simple poles P_i, P_j . Clearly, this saddle connection is twice shorter than the length of the periodic trajectories. The other boundary component is a saddle connection joining the zero P_l to itself. The total angle $k_l \pi$ at the singularity P_l is split by the separatrix loop into two sectors, such that the sector adjacent to the cylinder has angle π . The length of this saddle connection is the same as the length of the periodic trajectories in the cylinder.

IV. A “dumbbell”. In this last situation (see Figure 8) we again have a band of periodic trajectories. The boundary of the band is again composed of two generalized diagonals, but this time the first generalized diagonal joins the corner P_i with inner angle $k_i \frac{\pi}{2}$ to itself, and the second generalized diagonal joins the distinct corner P_j with inner angle $k_j \frac{\pi}{2}$ to itself. Both k_i, k_j are greater than or

equal to 3. The length of each of these two generalized diagonals is the same as the length of every periodic billiard trajectory in the band.

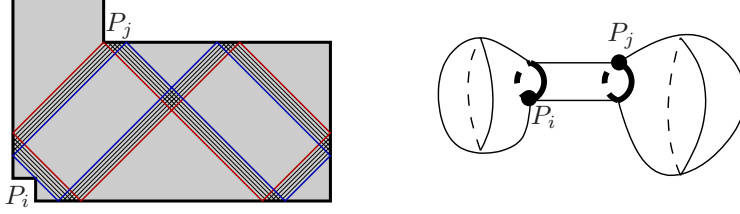


FIGURE 8. Type IV. On the left: a band of periodic trajectories, such that each of the two bounding generalized diagonals returns to the same corner. On the right: a “dumbbell” composed of two flat spheres joined by a cylinder. Each boundary component of the cylinder is a saddle connection joining a zero to itself.

For the associated flat metric on \mathbb{CP}^1 we get a cylinder filled with closed regular trajectories. On each of the boundary components of the cylinder we have a saddle connection joining the zero P_i (correspondingly P_j) to itself. The length of each of the two saddle connections is the same as the length of the periodic trajectories in the cylinder.

The following two Propositions explain why we distinguish these four particular types of configurations (see more details in §3.2 which discusses a homological interpretation of these statements).

Proposition 2.1. *Almost any flat surface S in any stratum $\mathcal{Q}_1(d_1, \dots, d_n)$ different from the pillowcase stratum $\mathcal{Q}_1(-1^4)$ does not have a single pair of parallel saddle connections different from the pairs involved in configurations of types I, II, III, IV.*

Proposition 2.1 is proved in §3.2. An analogous statement can be formulated for right-angled billiards.

Proposition 2.2. *For almost any right-angled billiard in any family $\mathcal{B}(k_1, \dots, k_n)$ the following property holds. Consider a pair of trajectories, where each trajectory is either a closed regular trajectory or a generalized diagonal. Suppose that these trajectories are not parallel to any side of the polygon. If some segment of the first trajectory is parallel to some segment of the second trajectory, then both trajectories make part of one of configurations I–IV described in 2.1.*

Proposition 2.2 mimics Proposition 7.4 in [EMZ03]; it is proved in §3.3.

Configurations of saddle connections. In addition to the type I–IV of a saddle connection, we may specify some extra combinatorial information, for example the indices (“names”) of all singularities involved. For saddle connections of type IV, where a cylinder is joining two spheres, we specify not only the zeroes P_i and P_j at the boundary components of the cylinder, but we also specify the subcollections $P_{i_1}, \dots, P_{i_{k_1}}$ and $P_{j_1}, \dots, P_{j_{k_2}}$ of numbered zeroes and poles which get to the first and to the second sphere correspondingly. We call this information the *configuration* of a saddle connection (or the configuration of saddle connections, when there are several saddle connections involved as in types III and IV). By convention, the

configuration of saddle connections includes its type. See also §3.2 for a homological interpretation of a configuration of saddle connections.

Configuration of a generalized diagonal. By the *configuration* of the generalized diagonal we mean the configuration of the associated saddle connections in \mathbb{CP}^1 described in §1.2.

2.2. Counting Theorems. By the notation

$$N(L) \sim cL^2$$

we mean as customary,

$$\lim_{L \rightarrow \infty} \frac{N(L)}{L^2} = c.$$

For technical reasons, we will need to consider “weak asymptotic formulas”

$$N(L) \sim cL^2$$

which means

$$\lim_{L \rightarrow \infty} \frac{1}{L} \int_0^L N(e^t) e^{-2t} dt = c.$$

The following theorem (which is a special case of results of [Ve98] and [EM00]) establishes a strong asymptotic formula for almost all flat surfaces in a stratum. By convention we always count *non-oriented* saddle connections and *non-oriented* closed flat geodesics.

Theorem 2.3. *For almost any flat surface S in any stratum $\mathcal{Q}(d_1, \dots, d_n)$ of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 the number $N_{\mathcal{C}}(S, L)$ of occurrences of saddle connections of length at most L and of fixed configuration \mathcal{C} , has quadratic asymptotics in L :*

$$N_{\mathcal{C}}(S, L) \sim c_{\mathcal{C}} \cdot \frac{\pi L^2}{\text{Area of } S}.$$

The constants $c_{\mathcal{C}}$ are called Siegel-Veech constants. They depend only on the configuration \mathcal{C} and on d_1, \dots, d_n . Their values are given in §4.

Theorem 2.3 is proved in §4.5. Note that Theorem 2.3 has no relation to billiards, it concerns only flat metrics on \mathbb{CP}^1 induced by meromorphic quadratic differentials with simple poles. In §1.2 we described how a right-angled billiard table Π canonically determines a meromorphic quadratic differential on \mathbb{CP}^1 . However, since the image of the resulting map $\mathcal{B}(k_1, \dots, k_n) \rightarrow \mathcal{Q}(k_1 - 2, \dots, k_n - 2)$ has measure 0 in $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$, results such as Theorem 2.3 do not immediately imply anything about right-angled billiards. Nevertheless, we have the following:

Theorem 2.4. *For almost any billiard table Π in any family $\mathcal{B}(k_1, \dots, k_n)$ of right-angled billiards the number $N_{\mathcal{C}}(\Pi, L)$ of occurrences of generalized diagonals of configuration \mathcal{C} and of length at most L has weak quadratic asymptotics in L :*

$$(2.1) \quad N_{\mathcal{C}}(\Pi, L) \sim \frac{c_{\mathcal{C}}}{4} \cdot \frac{\pi L^2}{\text{Area of the billiard table } \Pi},$$

where the constants $c_{\mathcal{C}}$ are the corresponding Siegel-Veech constants $c_{\mathcal{C}}$ for the stratum $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$ in Theorem 2.3.

Theorem 2.4 is proved in §6.1.

The factor of $\frac{1}{4}$ in (2.1) is explained as follows. Note that any generalized diagonal in the billiard table Π which is not parallel to one of the sides of Π canonically determines two symmetric saddle connections of the same type on the flat surface S glued from the two copies of Π , where the symmetry is the antiholomorphic involution, see Figure 3. Hence,

$$N_{\mathcal{C}}(\Pi, L) = \frac{1}{2} N_{\mathcal{C}}(S, L).$$

Note also, that by construction the area of S is twice the area of the billiard table Π .

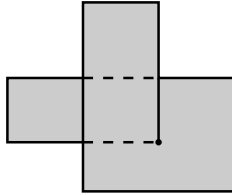


FIGURE 9. A helical billiard corresponds to the stratum $\mathcal{Q}(d, -1^{d+4})$.

Note that our billiard table does not need to be necessarily embeddable into the plane, say, we can consider a helical right-angled billiard as in Figure 9. More precisely, by a *right-angled billiard table* we call a topological disc endowed with a flat metric having the following properties. The flat metric is allowed to have isolated cone-type singularities in the interior of the disc with cone angles of the form $l_i\pi$, with $l_i \in \mathbb{N}$. The boundary of the disc is piecewise-geodesic in the flat metric, and the angles between the geodesic segments have the form $k_j\pi/2$, with $k_j \in \mathbb{N}$.

In fact, a stronger version of Theorem 2.4 holds:

Theorem 2.5. *Suppose Π is a billiard table from the family of right-angled billiards $\mathcal{B}(k_1, \dots, k_n)$. Furthermore, suppose Π is such that the flat surface S glued from two copies of Π does not belong to any proper $\mathrm{GL}(2, \mathbb{R})$ -invariant affine submanifold of the stratum $\mathcal{Q}(d_1, \dots, d_k)$. Then, for any choice I–IV of configuration \mathcal{C} , the weak asymptotic formula*

$$N_{\mathcal{C}}(\Pi, L) \sim \frac{c_{\mathcal{C}}}{4} \cdot \frac{\pi L^2}{\text{Area of the billiard table } \Pi}$$

holds, where $c_{\mathcal{C}}$ is the Siegel–Veech constant corresponding to the configuration \mathcal{C} in the stratum $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$ (as in Theorem 2.3).

Proof. The statement is an immediate corollary of Theorem 2.12 in [EMiMo]. \square

We note that a complete proof of Theorem 2.12 in [EMiMo] involves the measure classification theorem of [EMi] and is well over 200 pages long. In §3 and §6 of this paper, we give a much shorter and self-contained proof of the weaker statement (namely, of Theorem 2.4) which uses special features of right-angled billiards, namely Proposition 3.2.

In §6, we also prove the following L^1 -statement. Denote by $\mathcal{B}_1(k_1, \dots, k_n)$ the hypersurface of billiard tables of area one in the family $\mathcal{B}(k_1, \dots, k_n)$, and denote

by $\mu_{\mathcal{B}_1}$ the measure on $\mathcal{B}_1(k_1, \dots, k_n)$ induced from the Lebesgue measure $\mu_{\mathcal{B}}$ on $\mathcal{B}(k_1, \dots, k_n)$ to this hypersurface.

Theorem 2.6. *Let*

$$A \subset \mathcal{B}_1(k_1, \dots, k_n)$$

be a set of positive $\mu_{\mathcal{B}_1}$ -measure. Then for any configuration \mathcal{C} as $L \rightarrow \infty$,

$$(2.2) \quad \frac{1}{\mu_{\mathcal{B}_1}(A)} \int_A N_{\mathcal{C}}(\Pi, L) d\mu_{\mathcal{B}_1} \sim \frac{c_{\mathcal{C}}}{4} \cdot \frac{\pi L^2}{\text{Area of the billiard table } \Pi},$$

where $c_{\mathcal{C}}$ is the Siegel–Veech constant corresponding to the configuration \mathcal{C} in the stratum $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$.

Theorem 2.6 is proved in §6.2.

3. BILLIARDS IN RIGHT-ANGLED POLYGONS AND QUADRATIC DIFFERENTIALS

In §3.1 we describe the cohomological coordinates in a stratum of quadratic differentials. We proceed in §3.2 with a reminder of the notions of *homologous saddle connections* and a *configuration* of homologous saddle connections.

In §3.3 we analyze the canonical embedding of the space of (directional) right-angled billiards $\mathcal{B}(k_1, \dots, k_n)$ into the corresponding space $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$ of meromorphic quadratic differentials on \mathbb{CP}^1 . Namely, we prove in Proposition 3.2 that its image projects surjectively onto the unstable foliation of the Teichmüller geodesic flow, which allows us to apply certain ergodic techniques of hyperbolic dynamics not only to flat surfaces from $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$ but to billiards from $\mathcal{B}(k_1, \dots, k_n)$.

We complete §3 with a proof of Proposition 2.2.

3.1. Coordinates in a stratum of quadratic differentials. Consider a meromorphic quadratic differential ψ having zeroes of arbitrary multiplicities but only simple poles on \mathbb{CP}^1 . Let P_1, \dots, P_n be its singular points (zeros and simple poles). Consider the minimal branched double covering $p: \hat{S} \rightarrow \mathbb{CP}^1$ such that the induced quadratic differential $p^*\psi$ on the hyperelliptic surface \hat{S} is a square of an Abelian differential $p^*\psi = \omega^2$.

The zeros $\hat{P}_1, \dots, \hat{P}_N$ of the resulting Abelian differential ω correspond to the zeros of ψ in the following way: every zero $P \in \mathbb{CP}^1$ of ψ of odd order is a ramification point of the covering, so it produces a single zero $\hat{P} \in \hat{S}$ of ω ; every zero $P \in \mathbb{CP}^1$ of ψ of even order is a regular point of the covering, so it produces two zeros $\hat{P}^+, \hat{P}^- \in \hat{S}$ of ω . Every simple pole of ψ defines a branching point of the covering; this point is a regular point of ω .

Consider the subspace $H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$ of the relative homology of the cover with respect to the collection of zeroes $\{\hat{P}_1, \dots, \hat{P}_N\}$ of ω which is antiinvariant with respect to the induced action of the hyperelliptic involution. We are going to construct a basis in this subspace (in complete analogy with a usual basis of absolute cycles for a hyperelliptic surface).

We can always enumerate the singular points P_1, \dots, P_n of ψ in such a way that P_n is a simple pole. Choose now a simple oriented broken line P_1, \dots, P_{n-1} on \mathbb{CP}^1 joining consecutively all the singular points of ψ except the last one. For every arc $[P_i, P_{i+1}]$ of this broken line, $i = 1, \dots, n-2$, the difference of their two preimages defines a relative cycle in $H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$. By construction such a cycle is

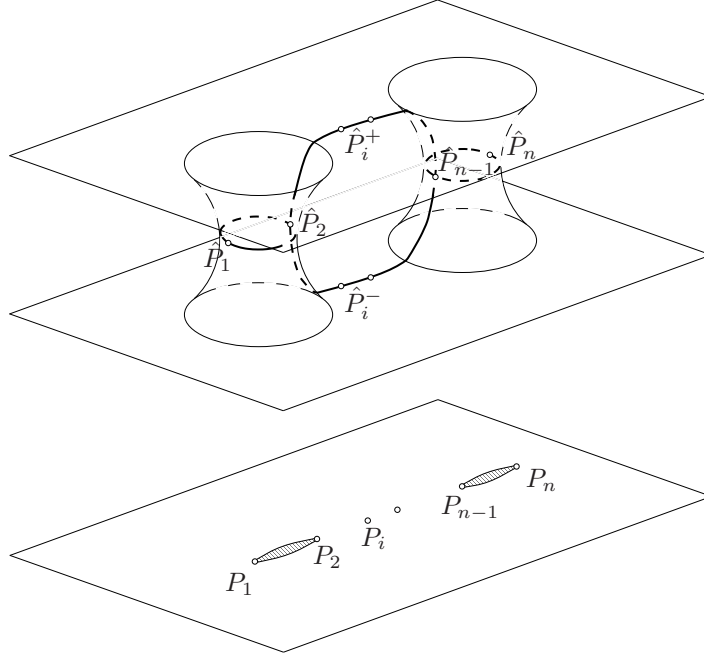


FIGURE 10. Basis of cycles in $H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$. Note that the cycle corresponding to the very last slit is omitted.

antiinvariant with respect to the hyperelliptic involution. It is immediate to see that the resulting collection of cycles forms a basis in $H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$.

Note that a preimage of a simple pole does not belong to the set $\hat{P}_1, \dots, \hat{P}_N$. Thus, a preimage of an arc $[P_i, P_{i+1}]$ having a simple pole as one of the endpoints does not define a cycle in $H_1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$. However, since a simple pole is always a branching point, the *difference* of the preimages of such arc is already a well-defined relative cycle in $H_1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$.

Let $\mathcal{Q}(d_1, \dots, d_n)$ be the ambient stratum for the meromorphic quadratic differential (\mathbb{CP}^1, ψ) . The subspace $H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{C})$ in the relative *cohomology* antiinvariant with respect to the natural involution defines local coordinates in the stratum. INSERT BRIEF SECTION ON $SL(2, \mathbb{R})$ ACTION

3.2. Homologous saddle connections. We follow the exposition in [MZ08] introducing the notions of a *rigid* collection of saddle connections and of *homologous* saddle connections. Consider a flat sphere S corresponding to a meromorphic quadratic differential (\mathbb{CP}^1, ψ) with at most simple poles. Any saddle connection on the flat sphere S persists under small deformations of S inside $\mathcal{Q}(\alpha)$. It might happen that any deformation of a given flat surface which shortens some specific saddle connection necessarily shortens some other saddle connections. We say that a collection $\{\gamma_1, \dots, \gamma_n\}$ of saddle connections is *rigid* if any sufficiently small deformation of the flat surface inside the stratum preserves the proportions $|\gamma_1| : |\gamma_2| : \dots : |\gamma_n|$ of the lengths of all saddle connections in the collection.

Consider the canonical double cover \hat{S} over S defined in §3.1. Given a saddle connection γ on S choose an orientation of γ and let γ', γ'' be its lifts to the

double cover \hat{S} endowed with the orientation inherited from γ . If $[\gamma'] = -[\gamma'']$ as cycles in $H_1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$ we let $[\hat{\gamma}] := [\gamma']$, otherwise we define $[\hat{\gamma}]$ as $[\hat{\gamma}] := [\gamma'] - [\gamma'']$. It immediately follows from the above definition that the cycle $[\hat{\gamma}]$ defined by a saddle connection γ is always *primitive* in $H_1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$ and belongs to $H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$.

Following [MZ08] we introduce the following

Definition 3.1. The saddle connections γ_1, γ_2 on a flat surface S defined by a quadratic differential q are *homologous* if $[\hat{\gamma}_1] = [\hat{\gamma}_2]$ in $H_1(\hat{S}, \hat{P}; \mathbb{Z})$ under an appropriate choice of orientations of γ_1, γ_2 . (The notion “homologous in the relative homology with local coefficients defined by the canonical double cover induced by a quadratic differential” is unbearably bulky, so we introduced an abbreviation “homologous”. We stress that the circumflex over the “h” is quite meaningful: as it is indicated in the definition, the corresponding cycles are homologous *on the double cover*.)

Note that since there is no canonical way to enumerate the preimages γ', γ'' of a saddle connection γ on the double cover, the cycle $[\hat{\gamma}]$ is defined only up to a sign, even when we fix the orientation of γ . Thus, γ_1 is homologous to γ_2 if and only if $[\hat{\gamma}_1] = \pm[\hat{\gamma}_2]$.

Proposition (H. Masur, A. Z.). *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. A collection $\gamma_1, \dots, \gamma_n$ of saddle connections on S is rigid if and only if all saddle connections $\gamma_1, \dots, \gamma_n$ are homologous.*

There is an obvious geometric test for deciding when saddle connections γ_1, γ_2 on a translation surface S are homologous: it is sufficient to check whether $S \setminus (\gamma_1 \cup \gamma_2)$ is connected or not (provided $S \setminus \gamma_1$ and $S \setminus \gamma_2$ are connected). It is slightly less obvious to check whether saddle connections γ_1, γ_2 on a flat surface S with nontrivial linear holonomy are homologous or not. In particular, a pair of closed saddle connections might be homologous in the usual sense, but not homologous; a pair of closed saddle connections might be homologous even if one of them represents a loop homologous to zero, and the other does not; finally, a saddle connection joining a pair of *distinct* singularities might be homologous to a saddle connection joining a singularity to itself, or joining another pair of distinct singularities. The following statement provides a geometric criterion for deciding when two saddle connections are homologous.

Proposition (H. Masur, A. Z.). *Let S be a flat surface corresponding to a meromorphic quadratic differential q with at most simple poles. Two saddle connections γ_1, γ_2 on S are homologous if and only if they have no interior intersections and one of the connected components of the complement $S \setminus (\gamma_1 \cup \gamma_2)$ has trivial linear holonomy. Moreover, if such a component exists, it is unique.*

Now everything is ready for the proof of Proposition 2.1.

Proof of Proposition 2.1. Configurations I and II involve a single saddle connection. Using the above criterion it is immediate to check that all saddle connections involved in configurations III and IV are homologous. Thus, these configurations are rigid, and we can find them on almost every flat surface in the stratum.

Theorem 2.2 in [Bo] applies general results from [MZ08] to classify all possible configurations of homologous saddle connections on \mathbb{CP}^1 , and shows that there are no such configurations different from types I–IV.

To complete the proof it remains to apply Proposition 4 from [MZ08] which claims that for almost every flat surface in any stratum, two saddle connections are parallel if and only if they are homologous. (This statement is proved following the lines of Proposition 7.4 in [EMZ03]; see also the analogous proof of Proposition 2.2 in §3.3 below. \square)

3.3. The subspace of billiards. Consider now the map

$$\mathcal{B}(k_1, \dots, k_n) \hookrightarrow \mathcal{Q}(k_1 - 2, \dots, k_n - 2).$$

In the chosen coordinates in $H_-^1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{C})$ the image of a directional billiard Π is presented by a point

$$(3.1) \quad \left(2 \int_{P_1}^{P_2} dz, \dots, 2 \int_{P_{n-2}}^{P_{n-1}} dz \right) = \left(2|P_1 P_2|e^{i\phi}, 2|P_2 P_3|e^{(k_2\pi)/2+i\phi}, \dots, 2|P_{n-3} P_{n-2}|e^{(k_2+\dots+k_{n-2})\pi/2+i\phi} \right).$$

The components of the projection of this vector to the $H_-^1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{R})$ are of the form

$$\pm 2 \sin(\phi) |P_i P_{i+1}| \quad \text{or} \quad \pm 2 \cos(\phi) |P_i P_{i+1}|$$

depending on the parity of $k_2 + \dots + k_i$. Thus, for ϕ different from an integer multiple of $\pi/2$ the composition map $T_*\mathcal{B} \rightarrow H_-^1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{R})$ is a surjective map. We have proved

Proposition 3.2. *Consider the canonical local embedding*

$$\mathcal{B}(k_1, \dots, k_n) \hookrightarrow \mathcal{Q}(k_1 - 2, \dots, k_n - 2).$$

For almost all directional billiards in $\mathcal{B}(k_1, \dots, k_n)$ the projection of the tangent space $T_\mathcal{B}(k_1, \dots, k_n)$ to the unstable subspace of the Teichmüller geodesic flow is a surjective map.*

Using standard ergodic theoretic arguments on unstable foliations (see, e.g., [Mar70, EMcM93]), we have the following:

Corollary 3.3. *Let $\mu_{\mathcal{B}}$ denote Lebesgue measure (arising from lengths of sides and the angle ϕ) on $\mathcal{B} = \mathcal{B}(k_1, \dots, k_n)$, and let $\mu_{\mathcal{Q}}$ denote Lebesgue measure on $\mathcal{Q}(k_1 - 2, \dots, k_n - 2)$. Then for $\mu_{\mathcal{B}}$ -almost every $\Pi \in \mathcal{B}$, Lebesgue almost every $\theta \in S^1$, we have, for any $f \in C_c(\mu_{\mathcal{Q}})$,*

$$(3.2) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(g_t r_{\theta} q_{\Pi}) dt = \frac{1}{\mu_{\mathcal{Q}}(\mathcal{Q})} \int_{\mathcal{Q}} f d\mu_{\mathcal{Q}}.$$

Moreover, for any set of $\mu_{\mathcal{B}}$ -positive measure $A \subset \mathcal{B}$, we have

$$(3.3) \quad \lim_{T \rightarrow \infty} \frac{1}{2\pi \mu_{\mathcal{B}}(A)} \int_A \int_0^{2\pi} f(g_T r_{\theta} q_{\Pi}) d\mu_{\mathcal{B}}(\Pi) = \frac{1}{\mu_{\mathcal{Q}}(\mathcal{Q})} \int_{\mathcal{Q}} f d\mu_{\mathcal{Q}}.$$

We complete this section with a proof of Proposition 2.2.

Proof of Proposition 2.2. By assumption we do not consider generalized diagonals and closed billiard trajectories parallel to the sides of the polygon. First note that without loss of generality we can consider only generalized diagonals: any closed regular trajectory makes part of a band which is bounded on both sides by a (chain of) generalized diagonals, see Figure 1.

Let $l_m = |P_m P_{m+1}|$ for $m = 1, \dots, n-2$. Recall that l_i are the independent coordinates in the space $\mathcal{B}(k_1, \dots, k_n)$. Unfolding the billiard along a generalized diagonal we see that every generalized diagonal (non parallel to one of the sides of the polygon) defines a relation

$$\frac{\sum a_i l_i}{\sum b_j l_j} = \tan(\phi),$$

where $0 < \phi < \pi/2$; the sum in the numerator is taken over the vertical sides of the polygon; the sum in the denominator is taken over the horizontal sides; and all a_i and b_j are integers. Since the second generalized diagonal has a segment going in the same direction ϕ , it also defines a relation

$$\frac{\sum c_i l_i}{\sum d_j l_j} = \tan(\phi),$$

where the sum in the numerator is taken over the vertical sides of the polygon; the sum in the denominator is taken over the horizontal sides; and all c_i and d_j are integers.

Each generalized diagonal determines a saddle connection γ on the corresponding flat sphere, which in turn defines a cycle $\pm\hat{\gamma} \in H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$. Moreover, up to appropriate choice of signs of the basic vectors in the basis from §3.1 the cycle corresponding to the first generalized diagonal has the form $\hat{c}_1 := \sum a_i \hat{\gamma}_i + \sum b_j \hat{\gamma}_j$ and the cycle corresponding to the second generalized diagonal has the form $\hat{c}_2 := \sum c_i \hat{\gamma}_i + \sum d_j \hat{\gamma}_j$.

Assume that the two generalized diagonals do not make part of any of configurations I–IV. By the result of Boissy [Bo] there are no configurations of $\hat{\text{homologous}}$ saddle connections on \mathbb{CP}^1 other than configurations I–IV. This implies that the corresponding saddle connections are not $\hat{\text{homologous}}$, and, hence, the cycles \hat{c}_1 and \hat{c}_2 are not proportional. This implies that the relation

$$\frac{\sum a_i l_i}{\sum b_j l_j} = \frac{\sum c_i l_i}{\sum d_j l_j}$$

is a nontrivial relation on coordinates l_1, \dots, l_{n-2} . Thus, the set, satisfying this condition, has measure zero. Taking a union over the countable collection of possible conditions (countable, because we have to consider all possible collections of integers a_i, b_j, c_i, d_j) we still get a set of measure zero. \square

4. VALUES OF THE SIEGEL-VEECH CONSTANTS

In this section, we derive formulas for the Siegel-Veech constant of each configuration of saddle connections. There are two kinds of formulas. The first kind expresses the Siegel-Veech constant as a ratio of volumes of strata, with explicit combinatorial coefficients. These formulas will be stated and proved in this section. The second kind of formula gives the Siegel-Veech constants as numbers (depending only on the stratum and the configuration). They are proved by plugging the expression (1.8) from Theorem 1.5 into the formula of the first kind. We also present

these formulas here; however, Theorem 1.5 will only be proved in §5. For this reason we have attempted to separate the formulas which depend on Theorem 1.5 from the formulas which do not.

The results obtained in this section are based on techniques developed in the papers [EM00], [EMZ03], and [MZ08] written in collaboration with H. Masur.

4.1. Normalization of the volume element. Recall that for any flat surface S in any stratum $\mathcal{Q}(d_1, \dots, d_k)$ we have a canonical ramified double cover $\hat{S} \rightarrow S$ such that the induced quadratic differential on the Riemann surface \hat{S} is a global square of a holomorphic Abelian differential. We have seen in §3.1 that the subspace $H_-^1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{C})$ antiinvariant with respect to the induced action of the hyperelliptic involution on relative cohomology provides local coordinates in the corresponding stratum $\mathcal{Q}(d_1, \dots, d_n)$ of quadratic differentials. We define a lattice in $H_-^1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{C})$ as the subset of those linear forms which take values in $\mathbb{Z} \oplus i\mathbb{Z}$ on $H_1^-(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$.

We define the volume element $d\mu$ on $\mathcal{Q}(d_1, \dots, d_k)$ as the linear volume element in the vector space $H_-^1(\hat{M}_g^2, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{C})$ normalized in such way that the fundamental domain of the above lattice has volume 1.

We warn the reader that for $N > 1$ this lattice is a proper sublattice of index 4^{N-1} of the lattice

$$H_-^1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{C}) \cap H^1(\hat{S}, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z} \oplus i\mathbb{Z}).$$

Indeed, if a flat surface S defines a lattice point for our choice of the lattice, then the holonomy vector along a saddle connection joining distinct singularities might be half-integer. (However, the holonomy vector along any *closed* saddle connection is still always integer.)

The choice of one or another lattice is a matter of convention. Our choice makes formulae relating enumeration of pillowcase covers to volumes simpler; see Appendix B. Another advantage of our choice is that the volumes of the strata $\mathcal{Q}(d, -1^{d+4})$ and of the hyperelliptic components of the corresponding strata of Abelian differentials are the same (up to the factors responsible for the numbering of zeroes and of simple poles).

Convention 4.1. Similar to the case of Abelian differentials we now choose a real hypersurface $\mathcal{Q}_1(d_1, \dots, d_k)$ in the stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ of flat surfaces of fixed area. We abuse notation by denoting by $\mathcal{Q}_1(d_1, \dots, d_k)$ the space of flat surfaces of area 1/2 (so that the canonical double cover has area 1).

The volume element $d\mu$ in the embodying space $\mathcal{Q}(d_1, \dots, d_k)$ induces naturally a volume element $d\mu_1$ on the hypersurface $\mathcal{Q}_1(d_1, \dots, d_k)$ in the following way. There is a natural \mathbb{C}^* -action on $\mathcal{Q}(d_1, \dots, d_k)$: having $\lambda \in \mathbb{C}^*$ we associate to the flat surface $S = (\mathbb{CP}^1, q)$ the flat surface

$$(4.1) \quad \lambda \cdot S := (\mathbb{CP}^1, \lambda^2 \cdot q).$$

In particular, we can represent any $S \in \mathcal{Q}(d_1, \dots, d_k)$ as $S = rS_{(1)}$, where $r \in \mathbb{R}_+$, and where $S_{(1)}$ belongs to the “hyperboloid”: $S_{(1)} \in \mathcal{Q}_1(d_1, \dots, d_k)$. Geometrically this means that the metric on S is obtained from the metric on $S_{(1)}$ by rescaling with linear coefficient r . In particular, vectors associated to saddle connections on $S_{(1)}$ are multiplied by r to give vectors associated to corresponding saddle connections on S . It means also that $\text{area}(S) = r^2 \cdot \text{area}(S_{(1)}) = r^2/2$, since $\text{area}(S_{(1)}) =$

1/2. We define the *volume element* $d\mu_1$ on the “hyperboloid” $\mathcal{Q}_1(d_1, \dots, d_k)$ by disintegration of the volume element $d\mu$ on $\mathcal{Q}(d_1, \dots, d_k)$:

$$(4.2) \quad d\mu = r^{2n-1} dr d\mu_1,$$

where

$$2n = \dim_{\mathbb{R}} \mathcal{Q}(d_1, \dots, d_k) = 2 \dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k) = 2(k-2).$$

Using this volume element we define the total *volume of the stratum* $\mathcal{Q}_1(d_1, \dots, d_k)$:

$$(4.3) \quad \text{Vol } \mathcal{Q}_1(d_1, \dots, d_k) := \int_{\mathcal{Q}_1(d_1, \dots, d_k)} d\mu_1.$$

For a subset $E \subset \mathcal{Q}_1(d_1, \dots, d_k)$ we let $C(E) \subset \mathcal{Q}_1(d_1, \dots, d_k)$ denote the “cone” based on E :

$$(4.4) \quad C(E) := \{S = rS_{(1)} \mid S_{(1)} \in E, 0 < r \leq 1\}.$$

Our definition of the volume element on $\mathcal{Q}_1(d_1, \dots, d_k)$ is consistent with the following normalization:

$$(4.5) \quad \text{Vol}(\mathcal{Q}_1(d_1, \dots, d_k)) = \dim_{\mathbb{R}} \mathcal{Q}(d_1, \dots, d_k) \cdot \mu(C(\mathcal{Q}_1(d_1, \dots, d_k))),$$

where $\mu(C(\mathcal{Q}_1(d_1, \dots, d_k)))$ is the total volume of the “cone” $C(\mathcal{Q}_1(d_1, \dots, d_k)) \subset \mathcal{Q}(d_1, \dots, d_k)$ measured by means of the volume element $d\mu$ on $\mathcal{Q}(d_1, \dots, d_k)$ defined above.

4.2. $SL(2, \mathbb{R})$ -action. There is an action of $SL(2, \mathbb{R})$ on the moduli space of quadratic differentials that preserves the stratification, and moreover, preserves ([M82, Ve82]) the measures on \mathcal{Q} and \mathcal{Q}_1 described above. Recall that a quadratic differential q determines (and is determined by) by an atlas of charts to \mathbb{C} whose transition maps are of the form $z \mapsto \pm z + c$. Since $SL(2, \mathbb{R})$ acts on \mathbb{C} via linear maps on \mathbb{R}^2 , given a quadratic differential q and a matrix $g \in SL(2, \mathbb{R})$, define the quadratic differential $g \cdot q$ via post-composition of charts with g . This action generalizes the action of $SL(2, \mathbb{R})$ on the space of (unit-area) flat tori $SL(2, \mathbb{R})/SL(2, \mathbb{Z})$. Note that $SL(2, \mathbb{R})$ preserves the area of the quadratic differential q , and in particular it preserves the level surface $\mathcal{Q}_1(d_1, \dots, d_k)$.

4.3. Strata of surfaces with marked points. In this section we shall also consider the strata $\mathcal{Q}_1(\alpha)$ of flat surfaces $S = (\mathbb{CP}^1, q)$ where we mark a regular point on the surface. Say, $\mathcal{Q}_1(2, 1^2, 0, -1^7)$ will denote the stratum of meromorphic quadratic differentials on \mathbb{CP}^1 with one zero of order 2, two zeroes of order 1 denoted by 1^2 , seven simple poles -1^7 , and one additional marked point: “zero of order 0”.

Let $\alpha = \{d_1, \dots, d_k\}$ be a set with multiplicities, where $d_i \in \{-1, 1, 2, 3, \dots\}$ for $i = 1, \dots, k$, and $\sum d_i = -4$. A stratum with a marked point $\mathcal{Q}(0, d_1, \dots, d_k)$ has the natural structure of a fiber bundle over the corresponding stratum without marked points $\mathcal{Q}(d_1, \dots, d_k)$. This bundle has the surface S (punctured at all singularities P_1, \dots, P_k) as a fiber over the “point” $S \in \mathcal{Q}(d_1, \dots, d_k)$. Clearly, the dimension of the “universal curve” $\mathcal{Q}(0, d_1, \dots, d_k)$ satisfies

$$(4.6) \quad \dim_{\mathbb{C}} \mathcal{Q}(0, d_1, \dots, d_k) = \dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k) + 1 = k - 1.$$

By convention we *always* mark a point on a flat torus. We denote the corresponding stratum $\mathcal{H}(0)$; it has dimension two: $\dim_{\mathbb{C}} \mathcal{H}(0) = 2$.

The natural measure on the stratum $\mathcal{Q}(0, d_1, \dots, d_k)$ with marked points disintegrates into a product measure, where the measure $d\mu_0$ along the fiber is proportional to the Lebesgue measure on S induced by the flat metric on S , and the measure

on the base $\mathcal{Q}(d_1 \dots, d_k)$ is the natural measure $d\mu_1$ on the corresponding stratum taken without marked points.

When the flat structure on S is defined by a *quadratic* differential the measure of the fiber S is different from the measure of the analogous fiber S with the flat structure defined by an *Abelian* differential. Namely, by Convention 4.1 the area of the surface S in terms of our flat metric defined by the quadratic differential is $1/2$. Note also, that a saddle connection γ joining a zero and a marked point and having half-integer linear holonomy $\pm \text{hol}(\gamma) \in \mathbb{R}^2$ defines an integer cycle $\hat{\gamma} \in H_1^-(\hat{M}_g^2, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{Z})$. Hence, our choice of the fundamental domain of the lattice in the relative cohomology $H_-^1(\hat{M}_g^2, \{\hat{P}_1, \dots, \hat{P}_N\}; \mathbb{C})$ described in §4.1 implies that the component $d\mu_0$ of the disintegrated measure along the fiber S is

$$(4.7) \quad d\mu_0 = 4dx dy,$$

i.e. 4 times the standard Lebesgue measure coming from the flat metric. This gives $\mu_0(S) = 2$ for the total measure of each fiber, which implies the following relation between the volumes of the strata:

$$(4.8) \quad \text{Vol } \mathcal{Q}_1(0, d_1, \dots, d_k) = 2 \text{Vol } \mathcal{Q}_1(d_1, \dots, d_k).$$

Recall that $v(0) = 2$, see (1.7); so this is coherent with formula (1.8) for the volume.

4.4. Volume of a stratum of disconnected flat surfaces. It will be convenient to consider the strata $\mathcal{Q}(\alpha') = \mathcal{Q}(\alpha'_a) \times \mathcal{Q}(\alpha'_b)$, of closed flat surfaces S having two components $S_a \sqcup S_b$ of prescribed types. Such strata play especially important role in the context of the *principal boundary* discussed in §4.6. In the consideration below each of α'_a, α'_b might contain an entry “0” or not. In other words, the strata $\mathcal{Q}(\alpha'_a), \mathcal{Q}(\alpha'_b)$ are allowed to have a marked point.

Convention 4.2. Using notation $\alpha' = \alpha'_a \sqcup \alpha'_b$ for the strata $\mathcal{Q}(\alpha')$ of disconnected surfaces we assume that we keep track of how α' is partitioned into α'_a and α'_b .

We shall need the expressions for the volume element and for the total volume of such strata. The corresponding expressions for the strata of Abelian differentials were obtained in §6.2 pp. 81–82 in [EMZ03]. Though the corresponding formula translates to the strata of quadratic differentials without any difficulties we present this simple calculation since it is very instructive in view of calculation of Siegel–Veech constants performed below.

We write $S_i = r_i S_i^{(1)}$, where $\text{area}(S_i^{(1)}) = \frac{1}{2}$; $i \in \{a, b\}$. Then $\text{area}(S_i) = r_i^2 \cdot \frac{1}{2}$. Let

$$n_i := \dim_{\mathbb{C}} \mathcal{Q}(\alpha'_i); \quad n := \dim_{\mathbb{C}} \mathcal{Q}(\alpha') = n_a + n_b.$$

Let $d\mu^a$ (correspondingly $d\mu^b$) be the volume element on the stratum $\mathcal{Q}(\alpha'_a)$ (correspondingly $\mathcal{Q}(\alpha'_b)$). Let $d\mu_1^a$ (correspondingly $d\mu_1^b$) be the hypersurface volume element on the “unit hyperboloid” $\mathcal{Q}_1(\alpha'_a)$ (correspondingly $\mathcal{Q}_1(\alpha'_b)$). We have

$$d\mu(S) = d\mu^a(S_a) \cdot d\mu^b(S_b) = r_a^{2n_a-1} r_b^{2n_b-1} dr_a dr_b d\mu_1^a d\mu_1^b.$$

Set

$$W = \text{Vol } \mathcal{Q}_1(\alpha'_a) \cdot \text{Vol } \mathcal{Q}_1(\alpha'_b).$$

Then,

$$\mu(C(\mathcal{Q}_1(\alpha'))) = W \cdot \int_{r_a^2 + r_b^2 \leq 1} r_a^{2n_a-1} r_b^{2n_b-1} dr_a dr_b = W \cdot \frac{1}{4} \frac{(n_a-1)!(n_b-1)!}{n!},$$

where we have left the computation of the integral over the disk as an exercise. Hence, applying (4.5) we get

$$(4.9) \quad \begin{aligned} \text{Vol } \mathcal{Q}_1(\alpha') &= 2n \cdot \mu(C(\mathcal{Q}_1(\alpha'))) = \\ &= \frac{1}{2} \cdot \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_a) - 1)! (\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_b) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') - 1)!} \cdot \text{Vol } \mathcal{Q}_1(\alpha'_a) \cdot \text{Vol } \mathcal{Q}_1(\alpha'_b). \end{aligned}$$

Repeating literally the same arguments we obtain the corresponding formula for the volume elements:

$$(4.10) \quad d\mu_1 = \frac{1}{2} \cdot \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_a) - 1)! (\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_b) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') - 1)!} \cdot d\mu_1^a d\mu_1^b.$$

4.5. Reduction to ergodic theory. In this section we recall the strategy given in [EM00] to obtain the quadratic asymptotics in Theorem 2.3.

Fix an unordered collection (d_1, \dots, d_n) of integers $d_i \in \mathbb{N} \cup \{-1\}$, $i = 1, \dots, n$, satisfying $\sum_{i=1}^n d_i = -4$, and let \mathcal{Q}_1 denote the stratum $\mathcal{Q}_1(d_1, \dots, d_n)$. Note that every such stratum is nonempty and connected. Let μ_1 denote the canonical $\text{PSL}(2, \mathbb{R})$ -invariant measure on \mathcal{Q}_1 . Fix a configuration \mathcal{C} as in §2.1. To each saddle connection we associate a holonomy vector in the Euclidean plane \mathbb{R}^2 having the same length and the same line direction as the saddle connection. By convention the configuration III is represented by the closed saddle connection joining a zero to itself (the holonomy vector associated to the partner saddle connection joining two simple poles is parallel but twice shorter). Since by convention the saddle connections are not oriented, the holonomy vector is defined up to a sign, so we actually consider a pair of opposite holonomy vectors $\pm \vec{v}$. Given a flat surface $S = (\mathbb{CP}^1, q) \in \mathcal{Q}_1$, let $V_{\mathcal{C}}(S)$ be the set of holonomy vectors of saddle connections whose configuration is \mathcal{C} . For any flat surface S the set $V_{\mathcal{C}}(S)$ is a discrete subset of \mathbb{R}^2 . We are interested in the asymptotics of the number

$$(4.11) \quad N_{\mathcal{C}}(S, L) = \frac{1}{2} |V_{\mathcal{C}}(S) \cap B(0, L)|,$$

of saddle connections of type \mathcal{C} on the flat surface S of length at most L . The weight $1/2$ in the above expression compensates the fact that each saddle connection is represented by two holonomy vectors $\pm \vec{v}$.

In the remainder of §6, the stratum \mathcal{Q} and the configuration \mathcal{C} are fixed. We will often omit \mathcal{C} from the notation, and we will use the abbreviated notation q for the flat surface $S = (\mathbb{CP}^1, q)$.

4.5.1. Siegel-Veech formulas. Given $f \in C_c(\mathbb{R}^2)$, define the *Siegel-Veech* transform $\hat{f} : \mathcal{Q}_1 \rightarrow \mathbb{R}$ by

$$(4.12) \quad \hat{f}(q) = \frac{1}{2} \sum_{v \in V_{\mathcal{C}}(q)} f(v).$$

We have the *Siegel-Veech formula* ([Ve98], Theorem 0.5). There is a constant (called the *Siegel-Veech* constant) $b_{\mathcal{C}}(\mathcal{Q})$ so that:

$$(4.13) \quad \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \hat{f}(q) d\mu_1(q) = b_{\mathcal{C}}(\mathcal{Q}_1) \int_{\mathbb{R}^2} f(x, y) dx dy.$$

Let

$$g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

$$r_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.$$

Let f be (a smoothed version of) the indicator function of the trapezoid \mathcal{T} defined by the points

$$(1, 1), (-1, 1), (1/2, 1/2), (-1/2, 1/2).$$

Note that the area of this trapezoid is $3/4$.

We then have, for $t \gg 0$, and any $v \in \mathbb{R}^2$ ([EM00], Lemma 3.4):

$$(4.14) \quad \frac{1}{2\pi} \int_0^{2\pi} f(g_t r_\theta v) d\theta \approx \begin{cases} \frac{e^{-2t}}{\pi} & e^t/2 \leq \|v\| \leq e^t \\ 0 & \text{otherwise} \end{cases}.$$

(See [EM00] for the exact meaning of \approx). Heuristically, the integral measures the proportion of angles θ so that $r_\theta v \in g_{-t}\mathcal{T}$. The trapezoid $g_{-t}\mathcal{T}$ has vertices at

$$(e^{-t}, e^t), (-e^{-t}, e^t), (e^{-t}/2, e^t/2), (-e^{-t}/2, e^t/2).$$

The range of (inverse) slopes is of size $2e^{-2t}$, and thus the length of the interval of θ 's satisfying $r_\theta v \in g_{-t}\mathcal{T}$ is also of size $2e^{-2t}$, if v has length in between $e^t/2$ and e^t , and zero otherwise. Dividing by 2π to get the proportion, we obtain (4.14). Combining (4.13) and (4.14), we obtain

$$(4.15) \quad \frac{e^{2t}}{2\pi} \int_0^{2\pi} \hat{f}(g_t r_\theta q) d\theta \approx \frac{1}{\pi} (N_C(q, e^t) - N_C(q, e^t/2)).$$

4.5.2. Equidistribution results. The equation (4.15) reduces the problem of studying

$$\lim_{t \rightarrow \infty} e^{-2t} N_C(q, e^t)$$

to that of studying the limiting behavior of

$$\frac{1}{2\pi} \int_0^{2\pi} \hat{f}(g_t r_\theta q) d\theta$$

Assuming this limit exists, and is equal to c , a geometric series calculation shows

$$\lim_{t \rightarrow \infty} e^{-2t} N_C(q, e^t) = \frac{4}{3} \pi c.$$

Assuming further that Lebesgue measure supported on the circles $\{g_t r_\theta q\}_{0 \leq \theta < 2\pi}$ converges, as $t \rightarrow \infty$, to the absolutely continuous $SL(2, \mathbb{R})$ -invariant measure μ_1 on \mathcal{Q}_1 , we would have that $c = \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \hat{f}(q) d\mu_1(q)$, and then using (4.13), we would obtain, since the area of the trapezoid is $3/4$,

$$\lim_{t \rightarrow \infty} e^{-2t} N_C(q, e^t) = \pi b_C(\mathcal{Q}).$$

In fact, this is the approach used in [EM00]. There, the key tool is a general ergodic theorem on $SL(2, \mathbb{R})$ -actions, proved by A. Nevo [Nevo] which shows

$$\lim_{t \rightarrow \infty} \int_0^{2\pi} \hat{f}(g_t r_\theta q) d\theta = \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \hat{f}(q) d\mu_1(q),$$

for almost every $q \in \mathcal{Q}$. However, since the set of billiards has measure 0, this does not yield any information about them. We will instead use Corollary 3.3 to obtain our time- and space- averaged results.

4.6. Siegel–Veech constants and the principal boundary of strata. In this section we present a strategy for evaluation Siegel–Veech constants. This strategy was successfully applied in [EMZ03] to compute all Siegel–Veech constants for all connected components of the strata of Abelian differentials. In this section we present the general scheme elaborated in [EMZ03] and developed in [MZ08]. In the further sections we adjust it to the concrete cases of configurations of saddle connections I–IV described in §2.1.

Fix a stratum $\mathcal{Q}(\alpha)$ of meromorphic quadratic differentials on \mathbb{CP}^1 , where $\alpha = \{d_1, \dots, d_k\}$. Consider a configuration \mathcal{C} of one of the types I–IV (in the case of general strata in higher genus it would be any configuration of homologous saddle connections). We have seen in §4.5 that to each flat surface $S \in \mathcal{Q}(\alpha)$ we can associate a discrete subset $V_{\mathcal{C}}(S) \subset \mathbb{R}^2$ of holonomy vectors of saddle connections whose configuration is \mathcal{C} . By construction the set $V_{\mathcal{C}}(S)$ is centrally symmetric with respect to the origin. To any function f with compact support on \mathbb{R}^2 formula (4.12) associates its *Siegel–Veech transform* \hat{f} defined on the stratum \mathcal{Q} . By definition (4.12), choosing the characteristic function χ_L of a closed disk of radius L centered at the origin of \mathbb{R}^2 as a function f , we get as $\hat{\chi}_L(S)$ the counting function $N_{\mathcal{C}}(S, L)$ of the number of saddle connections of type \mathcal{C} and of length at most L on the flat surface S defined by (4.11).

Applying Siegel–Veech formula (4.13) we obtain

$$(4.16) \quad \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \hat{\chi}_L(S) d\mu_1(S) = b_{\mathcal{C}}(\mathcal{Q}) \int_{\mathbb{R}^2} \chi_L(x, y) dx dy = b_{\mathcal{C}}(\mathcal{Q}) \cdot \pi L^2.$$

By the results of A. Eskin and H. Masur [EM00], for almost all flat surfaces S in the stratum \mathcal{Q}_1 one has

$$(4.17) \quad N_{\mathcal{C}}(S, L) = \hat{\chi}_L(S) \sim b_{\mathcal{C}} \cdot \pi L^2$$

with the same constant $b_{\mathcal{C}}$ as in (4.16).

Formula (4.16) can be applied to $\hat{\chi}_L$ for any value of L . In particular, instead of taking large L we can choose a very small $L = \varepsilon \ll 1$. The corresponding function $\hat{\chi}_{\varepsilon}(S)$ counts how many (collections of) ε -short saddle connections (closed geodesics) of the type \mathcal{C} we can find on a flat surface $S \in \mathcal{Q}$.

Consider a subset $\mathcal{Q}_1^{\varepsilon}(\mathcal{C}) \subset \mathcal{Q}_1$ of surfaces of area $1/2$ having a saddle connection shorter than ε . Consider a smaller subset $\mathcal{Q}_1^{\varepsilon, thin} \subset \mathcal{Q}_1^{\varepsilon}$ of those surfaces of area $1/2$ in \mathcal{Q}_1 which have at least two nonhomologous saddle connections of type \mathcal{C} and of length at most ε . Finally, define $\mathcal{Q}_1^{\varepsilon, thick}$ as the complement $\mathcal{Q}_1^{\varepsilon} - \mathcal{Q}_1^{\varepsilon, thin}$.

For the flat surfaces S outside of the subset $\mathcal{Q}_1^{\varepsilon}(\mathcal{C})$ there are no saddle connections of the type \mathcal{C} shorter than ε , so $\hat{\chi}_{\varepsilon}(S) = 0$ for such surfaces. For surfaces S from the subset $\mathcal{Q}_1^{\varepsilon, thick}(\mathcal{C})$ there is exactly one collection like this, so $\hat{\chi}_{\varepsilon}(S) = 1$. Finally, for the surfaces S from the remaining subset $\mathcal{Q}_1^{\varepsilon, thin}(\mathcal{C})$ one has $\hat{\chi}_{\varepsilon}(S) \geq 1$. A. Eskin and H. Masur have proved in [EM00] that though $\hat{\chi}_{\varepsilon}(S)$ might be large on $\mathcal{H}_1^{\varepsilon, thin}$ the measure of this subset is so small that

$$\int_{\mathcal{Q}_1^{\varepsilon, thin}(\mathcal{C})} \hat{\chi}_{\varepsilon}(S) d\mu_1 = o(\varepsilon^2)$$

and hence

$$\int_{\mathcal{Q}_1} \widehat{\chi}_\varepsilon(S) d\mu_1 = \text{Vol } \mathcal{Q}_1^{\varepsilon, \text{thick}}(\mathcal{C}) + o(\varepsilon^2).$$

This latter volume is almost the same as the volume $\text{Vol } \mathcal{Q}_1^\varepsilon(\mathcal{C})$, namely, by [MS93] one has $\text{Vol } \mathcal{Q}_1^\varepsilon(\mathcal{C}) = \text{Vol } \mathcal{Q}_1^{\varepsilon, \text{thick}}(\mathcal{C}) + o(\varepsilon^2)$. Taking into consideration that

$$\int_{\mathbb{R}^2} \chi_\varepsilon(x, y) dx dy = \pi \varepsilon^2$$

and applying Siegel–Veech formula (4.16) we get

$$\frac{\text{Vol } \mathcal{Q}_1^\varepsilon(\mathcal{C})}{\text{Vol } \mathcal{Q}_1} + o(\varepsilon^2) = b_{\mathcal{C}} \cdot \pi \varepsilon^2$$

which implies the following formula for the Siegel–Veech constant $b_{\mathcal{C}}$:

$$(4.18) \quad b_{\mathcal{C}} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi \varepsilon^2} \cdot \frac{\text{Vol } \mathcal{Q}_1^\varepsilon(\mathcal{C})}{\text{Vol } \mathcal{Q}_1}.$$

We complete this section by establishing an elementary relation between the Siegel–Veech constant $b_{\mathcal{C}}$ used in §6 and in §4.6 and the Siegel–Veech constant $c_{\mathcal{C}}$ used in §2. Recall that counting function (4.17)

$$N_{\mathcal{C}}(S, L) \sim b_{\mathcal{C}} \cdot \pi L^2$$

counts the number of saddle connections of type \mathcal{C} of length at most L on the flat surface $S \in \mathcal{Q}_1$. By convention 4.1 surfaces from \mathcal{Q}_1 have area $1/2$. Thus, applying the asymptotic formula (2.3) from Theorem 2.3 to the flat surface $S \in \mathcal{Q}_1$ we get

$$N_{\mathcal{C}}(S, L) \sim c_{\mathcal{C}} \cdot \frac{\pi L^2}{\text{Area of } S} = 2c_{\mathcal{C}} \cdot \pi L^2,$$

which implies that

$$(4.19) \quad b_{\mathcal{C}} = 2c_{\mathcal{C}}.$$

4.7. Principal boundary. When saddle connections of configuration \mathcal{C} are contracted by a continuous deformation, the limiting flat surface decomposes into one or several connected components represented by nondegenerate flat surfaces S'_1, \dots, S'_m . Let the initial surface S belong to a stratum $\mathcal{Q}(\alpha)$, where α is the set with multiplicities $\{d_1, \dots, d_k\}$. Let $\mathcal{Q}(\alpha'_j)$ be the stratum ambient for S'_j . The stratum $\mathcal{Q}(\alpha'_\mathcal{C}) = \mathcal{Q}(\alpha'_1) \sqcup \dots \sqcup \mathcal{Q}(\alpha'_m)$ of disconnected flat surfaces $S'_1 \sqcup \dots \sqcup S'_m$ is referred to as a *principal boundary* stratum of the stratum $\mathcal{Q}(\alpha)$. The principal boundary of any connected component of any stratum of Abelian differentials is described in [EMZ03]; the principal boundaries of strata of quadratic differentials are described in [MZ08].

The papers [EMZ03], [MZ08] also present the inverse construction. Consider any flat surface $S' := S'_1 \sqcup \dots \sqcup S'_m \in \mathcal{Q}(\alpha'_\mathcal{C})$ in the principal boundary of $\mathcal{Q}(\alpha)$; consider a vector $\vec{v} \in \mathbb{R}^2 \simeq \mathbb{C}$ such that $\|\vec{v}\| \leq \varepsilon$. One can reconstruct a flat surface $S \in \mathcal{Q}(\alpha)$ endowed with a collection of saddle connections of the type \mathcal{C} such that the linear holonomy along saddle connections is represented by $\pm \vec{v}$, and such that degeneration of S contracting the saddle connections in the collection gives the surface S' . When the configuration \mathcal{C} does not involve any cylinders, any flat surface $S' \in \mathcal{Q}_1$ and any holonomy vector \vec{v} define the surface $S \in \mathcal{Q}_1^\varepsilon(\mathcal{C})$, basically, up to some finite order ambiguity which can be explicitly computed. Moreover, the measure in $\mathcal{Q}_1^\varepsilon(\mathcal{C})$ disintegrates as the measure in $\mathcal{Q}_1(\alpha'_\mathcal{C})$ times the measure $d\mu_0$

in the space of parameters of the deformation. The latter space can be viewed as a finite cover of the space of holonomy vectors $\pm\vec{v}$, that is the quotient of the disk D_ε^2/\pm of radius ε over the central symmetry. As a result we get

$$(4.20) \quad \text{Vol}(\mathcal{Q}_1^\varepsilon(\mathcal{C})) = (\text{explicit factor}) \cdot \pi\varepsilon^2 \cdot \text{Vol } \mathcal{Q}_1(\alpha'_\mathcal{C}) + o(\varepsilon^2).$$

Thus, in order to compute the constant $b_\mathcal{C}$ by formula (4.18) it is sufficient to express the volume of $\text{Vol } \mathcal{Q}_1(\alpha')$ in terms of the volumes $\text{Vol } \mathcal{Q}_1(\alpha'_1), \dots, \text{Vol } \mathcal{Q}_1(\alpha'_m)$, and to compute the explicit factor, responsible for the fixed finite number of flat surfaces $S \in \mathcal{Q}_1^\varepsilon(\alpha)$ which correspond to a fixed flat surface $S' \in \mathcal{Q}(\alpha'_\mathcal{C})$ in the boundary stratum and to a fixed holonomy vector \vec{v} . The first problem is simple; the answer to this problem is given in §4.4; the second problem is solved for configurations I–IV in the remaining part of §4.

The situation for configurations which involve a cylinder is slightly more complicated, but similar to the previous one. In both cases, applying formula (4.18) and (4.20) we express the constant $b_\mathcal{C}$ as

$$(4.21) \quad b_\mathcal{C} = (\text{explicit combinatorial factor}) \cdot \frac{\prod_{j=1}^k \text{Vol } \mathcal{Q}_1(\alpha'_j)}{\text{Vol}(\mathcal{Q}_1(\alpha))}.$$

4.8. Surgeries on a flat surface. Consider a flat surface $S' \in \mathcal{Q}_1(\alpha'_\mathcal{C})$ in a stratum of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 , possibly with a marked point. Fix some zero, or a simple pole (or the marked regular point) P_i . Consider a vector $\pm\vec{v} \in \mathbb{R}^2$, defined up to reversing the direction. Assume that \vec{v} is much shorter than the shortest saddle connection on S .

The papers [EMZ03] and [MZ08] describe how to perform a small deformation of the surface S' breaking up the chosen singularity P_i of degree d_i into two singularities P'_i, P''_i of any two prescribed degrees d'_i and d''_i satisfying the relation $d'_i + d''_i = d_i$, where $d_i, d'_i, d''_i \in \{-1, 0, 1, 2, \dots\}$. The deformation can be performed in such way that the holonomy vector of the resulting tiny saddle connection joining the newborn singularities P'_i, P''_i is exactly $\pm\vec{v}$. This deformation is described in details in sections 8.1–8.2 in [EMZ03] and in section 6.3 in [MZ08]. When at least one of d'_i, d''_i is even, the deformation is local: it does not change the metric outside of a small neighborhood of P_i and it does not change the area of the flat surface. When both d'_i, d''_i are odd the deformation involves some arbitrariness and involves some small change of the area of the flat surface. A discussion in the original papers [EMZ03] and [MZ08] explains why both issues might be neglected in our calculations.

The cone angles at the distinguished singularity is equal to $\pi(d_i + 2)$. Thus, there are $(d_i + 2)$ geodesic rays in linear direction $\pm\vec{v}$ adjacent to P_i . Take a small disk D_ε^2 of radius ε centered in the origin and consider its quotient D_ε^2/\pm over the action of central symmetry. Letting the vector $\pm\vec{v}$ vary in D_ε^2/\pm and taking care of normalization (4.7) of the measure $d\mu_0$ on D_ε^2/\pm we get a set of parameters of measure

$$(4.22) \quad (d + 2) \cdot 4 \cdot \frac{\pi\varepsilon^2}{2} = 2(d + 2) \cdot \pi\varepsilon^2.$$

For this configuration the “(explicit factor)” in (4.20) equals $2(d + 2)$.

Consider now a particular case, when one of the newborn singularities P'_i, P''_i , say, P''_i is a simple pole. Since $d'_i + d''_i = d_i \geq -1$, the singularities P'_i, P''_i cannot be simple poles simultaneously. Making a slit along the short saddle connection joining

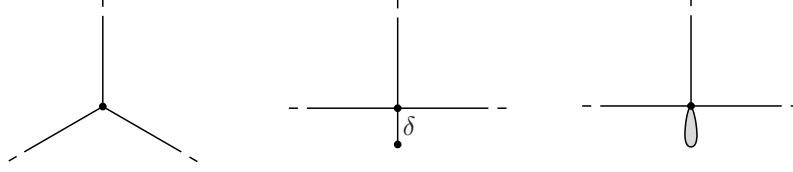


FIGURE 11. Breaking up a zero into two. In the particular case, when one of the newborn singularities is a simple pole, we can slit along the resulting saddle connection of length δ to get a surface with geodesic boundary of length 2δ .

P'_i to P''_i we create a surface \hat{S} with geodesic boundary. Note that the cone angle at the singularity P''_i was π . This means, that after opening up a slit, the point P''_i becomes a regular point of the boundary of \hat{S} , see Figure 11. In other words, the boundary of \hat{S} corresponds to a single closed geodesic with linear holonomy $\pm \vec{v}$.

Parallelogram construction. In order to construct the subset $\mathcal{Q}_1^\varepsilon(\mathcal{C})$ corresponding to configuration II, we need another surgery. Given a flat surface $S' \in \mathcal{Q}_1(\alpha')$ in a stratum of meromorphic quadratic differentials with at most simple poles on \mathbb{CP}^1 , given a pair of singularities P', P'' on S' and given a short vector $\pm \vec{v} \in \mathbb{R}^2$, we construct a surface with two boundary components creating a pair of small holes adjacent to the chosen singularities P', P'' . The surgery is performed in such way that the holes have geodesic boundary with linear holonomy $\pm \vec{v}$. Let d', d'' be the degrees of singularities P', P'' respectively. The corresponding cone angles are $\pi(d' + 2)$ and $\pi(d'' + 2)$. Thus, there are $(d' + 2)$ geodesic rays in linear direction $\pm \vec{v}$ adjacent to P' and $(d'' + 2)$ geodesic rays in linear direction $\pm \vec{v}$ adjacent to P'' .

The corresponding surgery is described in section 12.2 in [EMZ03] and in section 6.1 in [MZ08] as the “*parallelogram construction*”. This is a nonlocal construction, so it is not canonical, and it changes slightly the area of the surface. Up to this ambiguity (which can be neglected in our computations as explained in [EMZ03] and in [MZ08]), given the data as above, there are $(d' + 2)(d'' + 2)$ ways to construct the described surface with boundary \hat{S} . Take a small disk D_ε^2 of radius ε centered in the origin and consider its quotient D_ε^2/\pm over the action of central symmetry. Let the vector $\pm \vec{v}$ vary in D_ε^2/\pm . Note that in the contrary to the previous case, the saddle connection is now *closed*. Thus the measure along the fiber has the form

$$d\mu_0 = dx dy$$

and not the form (4.7) as before. This implies that for this configuration the set of parameters of deformation having holonomy vectors in D_ε^2/\pm has the measure

$$(4.23) \quad (d' + 2)(d'' + 2) \cdot \frac{\pi \varepsilon^2}{2}.$$

For this configuration the “(explicit factor)” in (4.20) equals $\frac{(d' + 2)(d'' + 2)}{2}$.

4.9. Type I: A simple saddle connection joining a fixed zero to a fixed pole or to a distinct fixed zero. Now we finally pass to explicit computation of the Siegel–Veech constants following the strategy described above.

Throughout the rest of this section $\mathcal{Q}(d_1, \dots, d_k)$ denotes any stratum of meromorphic quadratic differentials with at most simple poles on \mathcal{P} different from the stratum $\mathcal{Q}(-1^4)$ of pillowcases.

Theorem 4.3. *For the configuration \mathcal{C} of saddle connections of type I, i.e. for saddle connections joining a fixed pair P_i, P_j of distinct singularities of orders d_i, d_j , the Siegel–Veech constant $c_{\mathcal{C}}$ is expressed as follows:*

$$(4.24) \quad c_{\mathcal{C}} = (d_i + d_j + 2) \frac{\text{Vol } \mathcal{Q}_1(d_i + d_j, d_1, d_2, \dots, \widehat{d}_i, \dots, \widehat{d}_j, \dots, d_k)}{\text{Vol } \mathcal{Q}_1(d_1, \dots, d_k)}.$$

After plugging in Theorem 1.5, we get:

$$(4.25) \quad c_{\mathcal{C}} = \frac{(d_i + d_j + 2)!! (d_i + 1)!! (d_j + 1)!!}{(d_i + d_j + 1)!! d_i!! d_j!!} \cdot \begin{cases} \frac{2}{\pi^2} & \text{when both } d_i, d_j \text{ are odd} \\ \frac{1}{2} & \text{otherwise} \end{cases}$$

Proof of (4.24). The principal boundary $\mathcal{Q}_1(\alpha'_{\mathcal{C}})$ for this particular configuration \mathcal{C} is obtained by collapsing the saddle connection joining singularities of degrees d_i and d_j . This operation merges two singularities to a single one of degree $d = d_i + d_j$. Thus,

$$\alpha'_{\mathcal{C}} = \{d_i + d_j, d_1, d_2, \dots, \widehat{d}_i, \dots, \widehat{d}_j, \dots, d_k\}.$$

By (4.20)

$$\text{Vol}(\mathcal{Q}_1^{\varepsilon}(\mathcal{C})) = (\text{explicit factor}) \cdot \pi \varepsilon^2 \cdot \text{Vol } \mathcal{Q}_1(\alpha'_{\mathcal{C}}) + o(\varepsilon^2).$$

where the “(explicit factor) $\cdot \pi \varepsilon^2$ ” in formula (4.20) stands for the measure of the space of parameters of deformation corresponding to holonomy vectors in D_{ε}^2/\pm . This measure was computed in (4.22). Thus, we can rewrite (4.20) as

$$\text{Vol}(\mathcal{Q}_1^{\varepsilon}(\mathcal{C})) = 2(d+2) \cdot \pi \varepsilon^2 \cdot \text{Vol } \mathcal{Q}_1(d_i + d_j, d_1, d_2, \dots, \widehat{d}_i, \dots, \widehat{d}_j, \dots, d_k) + o(\varepsilon^2).$$

Applying (4.18) and (4.19) to the above expression we obtain (4.24). \square

We are ready to give a proof of Theorem 1.2 (based on (4.25) which would be proved in §5).

Proof of Theorem 1.2. Let $\mathcal{Q}(d_1, \dots, d_k) = \mathcal{Q}(1, -1^5)$. Let $d_i = 1, d_j = -1$. Applying (4.25) we get $c_I = 8/\pi^2$. Applying Theorem 2.4 to the L-shaped billiard as in Figure 2 we get the coefficient $\frac{2}{\pi}$ in the weak asymptotics of the number of generalized diagonals joining a fixed corner with angle $\frac{\pi}{2}$ with the corner with angle $\frac{3\pi}{2}$, and thus prove formula 1.2 and Theorem 1.2. \square

4.10. Type II: A simple saddle connection joining a zero to itself. The configuration \mathcal{C} of type II consists of a single separatrix loop emitted from a fixed zero P_i of order d_i such that the total angle $(d_i + 2)\pi$ at the singularity P_i is split by the separatrix loop into two sectors having the angles $(d'_i + 3)\pi$ and $(d''_i + 3)\pi$. We assume that $d'_i, d''_i \geq -1$, so we *do not* have any cylinders filled with periodic geodesics for this configuration. The angles satisfy the natural relation

$$d'_i + d''_i = d_i - 4 \quad d'_i, d''_i \geq -1$$

which implies, in particular, that $d_i \geq 2$.

Our saddle connection separates the original surface S into two parts. Let $P_{i_1}, \dots, P_{i_{k_1}}$ be the list of singularities (zeroes and poles) which belong to the first part and let $P_{j_1}, \dots, P_{j_{k_2}}$ be the list of singularities (zeroes and poles) which belong to the second part. This information is part of the configuration of this saddle connection.

We assume that the initial surface S does not have any marked points; as usual we denote by d_n the order of the singularity P_n . The set with multiplicities $\{d_1, \dots, d_k\}$ representing the orders of all singularities (zeroes and poles) on S can be obtained as a disjoint union of the following subsets:

$$\{d_1, \dots, d_k\} = \{d_{i_1}, \dots, d_{i_{k_1}}\} \sqcup \{d_{j_1}, \dots, d_{j_{k_2}}\} \sqcup \{d_i\}$$

Theorem 4.4. *The Siegel–Veech constant c_C for this configuration is expressed as follows:*

$$(4.26) \quad c_C = \frac{(d'_i + 2)(d''_i + 2)}{8} \cdot \frac{(\dim_{\mathbb{C}} \mathcal{Q}(d'_i, d_{i_1}, \dots, d_{i_{k_1}}) - 1)! (\dim_{\mathbb{C}} \mathcal{Q}(d''_i, d_{j_1}, \dots, d_{j_{k_2}}) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(d_1, d_2, \dots, d_k) - 2)!} \cdot \frac{\text{Vol } \mathcal{Q}_1(d'_i, d_{i_1}, \dots, d_{i_{k_1}}) \cdot \text{Vol } \mathcal{Q}_1(d''_i, d_{j_1}, \dots, d_{j_{k_2}})}{\text{Vol } \mathcal{Q}_1(d_1, \dots, d_k)}$$

After plugging in Theorem 1.5 we get:

$$(4.27) \quad c_C = \frac{1}{8} \cdot \frac{(d'_i + 2)!! (d''_i + 2)!! (d_i + 1)!!}{(d'_i + 1)!! (d''_i + 1)!! d_i!!} \cdot \frac{(k_1 - 2)! (k_2 - 2)!}{(k - 4)!} \cdot \begin{cases} 1 & \text{when both } d'_i, d''_i \\ & \text{are odd} \\ \frac{4}{\pi^2} & \text{otherwise} \end{cases}$$

Proof of (4.26). Let

$$\alpha'_a := \{d'_i, d_{i_1}, \dots, d_{i_{k_1}}\} \quad \alpha'_b := \{d''_i, d_{j_1}, \dots, d_{j_{k_2}}\} \quad \alpha'_C := \alpha'_a \sqcup \alpha'_b.$$

Contracting a saddle connection of type II and detaching the resulting singular flat surface into two components we get a disconnected flat surface $S' = S'_a \sqcup S'_b$, where $S' \in \mathcal{Q}(\alpha'_C)$. The stratum of disconnected surfaces $\mathcal{Q}(\alpha'_C)$ is the principal boundary for configuration II. By (4.20)

$$\text{Vol}(\mathcal{Q}_1^\varepsilon(C)) = (\text{explicit factor}) \cdot \pi \varepsilon^2 \cdot \text{Vol } \mathcal{Q}_1(\alpha'_C) + o(\varepsilon^2).$$

By (4.9) we have

$$\text{Vol } \mathcal{Q}_1(\alpha'_C) = \frac{1}{2} \cdot \frac{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_a) - 1)! (\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_b) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_C) - 1)!} \cdot \text{Vol } \mathcal{Q}_1(\alpha'_a) \cdot \text{Vol } \mathcal{Q}_1(\alpha'_b).$$

Note that by definition $\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_C) = \dim_{\mathbb{C}} \mathcal{Q}(\alpha'_a) + \dim_{\mathbb{C}} \mathcal{Q}(\alpha'_b)$. Hence

$$\dim_{\mathbb{C}} \mathcal{Q}(\alpha'_C) = ((k_1 + 1) - 2) + ((k_2 + 1) - 2) = (k_1 + k_2) - 2 = k - 3 = \dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2.$$

The “(explicit factor) $\cdot \pi \varepsilon^2$ ” in formula (4.20) stands for the measure of the space of parameters of deformation corresponding to holonomy vectors in D_ε^2/\pm . For

configuration \mathcal{C} of type II this measure was computed in (4.23). Thus, we can rewrite (4.20) as

$$\begin{aligned} \text{Vol}(\mathcal{Q}_1^\varepsilon(\mathcal{C})) &= \frac{(d' + 2)(d'' + 2)}{2} \cdot \pi \varepsilon^2. \\ &\frac{1}{2} \cdot \frac{(\dim_{\mathbb{C}} \mathcal{Q}(d'_i, d_{i_1}, \dots, d_{i_{k_1}}) - 1)! (\dim_{\mathbb{C}} \mathcal{Q}(d''_i, d_{j_1}, \dots, d_{j_{k_2}}) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(d_1, d_2, \dots, d_k) - 2)!} \cdot \\ &\quad \text{Vol } \mathcal{Q}_1(d'_i, d_{i_1}, \dots, d_{i_{k_1}}) \cdot \text{Vol } \mathcal{Q}_1(d''_i, d_{j_1}, \dots, d_{j_{k_2}}). \end{aligned}$$

Applying (4.18) and (4.19) to the above expression we obtain (4.26). \square

4.11. A “pocket”, i.e. a cylinder bounded by a pair of poles. Consider a configuration \mathcal{C} of type III where we have a single cylinder filled with closed regular geodesics, such that the cylinder is bounded by a saddle connection joining a fixed pair of simple poles P_{j_1}, P_{j_2} on one side and by a separatrix loop emitted from a fixed zero P_i of order $d_i \geq 1$ on the other side. This information is considered to be part of the configuration. By convention, the affine holonomy associated to this configuration corresponds to the closed geodesic and *not* to the saddle connection joining the two simple poles. (Such a saddle connection is twice as short as the closed geodesic.)

Theorem 4.5. *The Siegel–Veech constant $c_{\mathcal{C}}$ for this configuration is expressed as follows:*

$$(4.28) \quad c = \frac{d_i}{2(\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k) - 2)} \cdot \frac{\text{Vol } \mathcal{Q}_1(d_1, d_2, \dots, d_i - 2, \dots, d_k)}{\text{Vol } \mathcal{Q}_1(d_1, \dots, d_i, \dots, d_k)}.$$

After plugging in Theorem 1.5, we get

$$(4.29) \quad c_{\mathcal{C}} = \frac{d_i + 1}{2(k - 4)} \cdot \frac{1}{\pi^2}.$$

Proof of (4.28). Let $\alpha'_{\mathcal{C}} = \{d_1, \dots, d_{i-1}, d_i - 2, d_{i+1}, \dots, d_k\}$. Consider a configuration of type III with a short saddle connection γ joining a zero of degree d_i to itself. Contracting γ we get a flat surface S' in the principal boundary stratum $\mathcal{Q}(\alpha'_{\mathcal{C}})$.

To go backwards, we need to create a hole on S' with geodesic boundary having holonomy $\pm \vec{v}$ and attach a cylindrical “pocket” to this hole; see the right picture in Figure 7. The cone angle at the singularity P_i of degree $(d_i - 2)$ is $\pi \cdot d_i$. Thus, having a surface $S' \in \mathcal{Q}_1(\alpha'_{\mathcal{C}})$ and a vector $\pm \vec{v} \in D_{\varepsilon}^2 / \pm$ there are d_i rays in line direction $\pm \vec{v}$ adjacent to the singularity P_i .

Note, however, that now a deformation involves not only the surface S' from the principal boundary and a holonomy vector $\pm \vec{v}$, but also additional parameters describing the geometry of the “pocket”. Geometrically, a “pocket” is equivalent to a flat cylinder endowed with a distinguished line direction and with a marked point on each of the boundary components. Thus, in addition to the holonomy vector $\pm \vec{v}$ representing the waist curve, it is parameterized by the *height* h of the cylinder and by the *twist* t of the cylinder, $0 \leq t < |\vec{v}|$. Parameters h and t record the information about the holonomy along a saddle connection joining the zero P_i on one side of the cylinder to one of the simple poles, say, P_{j_1} on the other side of the cylinder. The flat area of a “pocket” $T(\pm \vec{v}, h, t)$ equals $|\vec{v}| \cdot h$.

The measure $d\mu$ in $\mathcal{Q}^{\varepsilon, thick}(\alpha)$ disintegrates into the product measure $d\mu'$ on $\mathcal{Q}(\alpha')$ and the measure $d\nu$ on the “space of pockets” \mathcal{R} ,

$$d\nu(S) = d\nu(S') \cdot d\nu(T).$$

The parameter $\pm\vec{v}$ corresponds to the holonomy along a *closed* saddle connection, while the parameters (h, t) correspond to holonomy along a saddle connection joining distinct singularities. Hence, the resulting measure on the space of parameters defining a “pocket” is

$$d\nu(T) = d\vec{v} \cdot 4dhdt.$$

Following Convention 4.1 we denote by \mathcal{R}_1 the hypersurface of pockets of area $\frac{1}{2}$. Let $S \in \mathcal{Q}_1(\alpha)$. We denote by $rS \in \mathcal{H}(\alpha)$ the surface proportional to the initial one with the coefficient r ; in particular, $area(rS) = r^2/2$, see Convention 4.1. We use similar notations $r_S S'$ and $r_T T$ for surfaces from $\mathcal{Q}(\alpha')$ and from \mathcal{R} correspondingly. We recall that the volume elements in the strata and the area elements on the corresponding “unit hyperboloids” are related as follows, see (4.2):

$$\begin{aligned} d\mu &= r^{2n-1} dr d\mu_1, & \text{where } n &= \dim_{\mathbb{C}} \mathcal{Q}(\alpha) = 2(k-2) \\ d\mu &= r_S^{2n_S-1} dr_S d\mu'_1, & \text{where } n_S &= \dim_{\mathbb{C}} \mathcal{Q}(\alpha') = 2(k-4) \\ d\nu &= r_T^{2n_T-1} dr_T d\nu_1, & \text{where } n_T &= \dim_{\mathbb{C}} \mathcal{R} = 4. \end{aligned}$$

Let $S' \in \mathcal{Q}_1(\alpha')$. Consider a surface $r_S S'$, where $0 < r_S \leq 1$; it has area $r_S^2/2$. Define $\Omega(\varepsilon, r_S) \subset \mathcal{R}$ to be the set of pockets, such that performing an appropriate surgery to $r_S S'$ and pasting in a “pocket” from $\Omega(\varepsilon, r_S)$ we get a surface $S \in C(\mathcal{Q}_1^{\varepsilon}(\alpha))$. Ignoring a negligible change of the area of the surface $r_S S'$ after creating a hole, we get the following two constraints. The first constraint imposes the bound on the area $r_T^2/2$ of a pocket $r_T T$, where $T \in \mathcal{R}_1$: the total area of the compound surface S should be at most $1/2$, so $r_S^2 + r_T^2 \leq 1$. The second constraint imposes a bound on the length of the waist curve of the cylinder: after rescaling proportionally the compound surface S to let it have area $\frac{1}{2}$ we should get a waist curve of length at most ε . Thus, the waist curve of the original cylinder should be at most $\varepsilon \sqrt{r_S^2 + r_T^2}$. Clearly, the set $\Omega(\varepsilon, r_S)$ does not depend on the particular surface $r_S S' \in \mathcal{Q}(\alpha')$, but only on the parameters r_S and ε .

We have seen that there are d_i rays in line direction $\pm\vec{v}$ adjacent to the singularity P_i . Using the above notations we can represent the volume of a cone in $\mathcal{Q}_1(\alpha)$ over $\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C})$ (see (4.4) for the definition of a *cone*) as

$$(4.30) \quad \mu(C(\mathcal{Q}_1^{\varepsilon}(\alpha, \mathcal{C}))) = d_i \cdot \text{Vol } \mathcal{Q}_1(\alpha') \cdot \int_0^1 \nu_T(\Omega(\varepsilon, r_S)) r_S^{2n_S-1} dr_S + o(\varepsilon^2).$$

Denote by $Cusp(\varepsilon)$ the volume of the ε -thin part of the “unit hyperboloid” in the space of “pockets”:

$$Cusp(\varepsilon) := \text{Vol } \mathcal{R}_1^{\varepsilon}.$$

From the definition of the subset $\Omega(\varepsilon, r_S)$ it immediately follows that its volume is expressed by the following integral

$$(4.31) \quad \nu_T(\Omega(\varepsilon, r_S)) = \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} \cdot Cusp\left(\frac{\varepsilon \cdot \sqrt{r_S^2 + r_T^2}}{r_T}\right) dr_T$$

Thus, we need to evaluate the following integral

$$(4.32) \quad \mu(C(\mathcal{Q}^\varepsilon(\alpha, \mathcal{C}))) = d_i \cdot \text{Vol}(\mathcal{Q}_1(\alpha')).$$

$$\int_0^1 r_S^{2n_S-1} dr_S \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} \cdot \text{Cusp} \left(\frac{\varepsilon \cdot \sqrt{r_S^2 + r_T^2}}{r_T} \right) dr_T + o(\varepsilon^2),$$

Lemma 4.6.

$$\text{Cusp}(\varepsilon) = 2\pi\varepsilon^2.$$

Proof. We first evaluate the volume $\nu(C(\mathcal{R}_1^\varepsilon))$ of the corresponding cone. Pockets belonging to this cone are described by the following conditions:

$$\begin{cases} h \cdot |\vec{v}| \leq 1/2 \\ |\vec{v}| \leq \varepsilon \cdot \sqrt{2h \cdot |\vec{v}|}. \end{cases}$$

Hence

$$\nu(C(\mathcal{R}_1^\varepsilon)) = \int_{D_\varepsilon^2/\pm} d\vec{v} \int_{w/(2\varepsilon^2)}^{1/(2w)} 2dh \int_0^w 2dt = 4\pi \int_0^\varepsilon w \left(\frac{1}{2w} - \frac{w}{2\varepsilon^2} \right) w dw = \frac{\pi\varepsilon^2}{2},$$

where $w = |\vec{v}|$. It remains to apply (4.5):

$$\nu(C(\mathcal{R}_1^\varepsilon)) = \dim_{\mathbb{R}} \mathcal{R} \cdot \text{Vol}(\mathcal{R}_1^\varepsilon)$$

and to note that $\dim_{\mathbb{R}} \mathcal{H}(0) = 4$. □

Having found the expression

$$\text{Cusp} \left(\frac{\varepsilon \cdot \sqrt{r_S^2 + r_T^2}}{r_T} \right) = 2\pi\varepsilon^2 \cdot \frac{r_S^2 + r_T^2}{r_T^2}$$

we can rewrite the integral (4.32) as

$$(4.33) \quad \nu(C(\mathcal{H}^\varepsilon(\alpha, \mathcal{C}))) = d_i \cdot \text{Vol} \mathcal{Q}_1(\alpha').$$

$$2\pi\varepsilon^2 \cdot \int_0^1 r_S^{2n_S-1} dr_S \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} \cdot \frac{r_S^2 + r_T^2}{r_T^2} dr_T + o(\varepsilon^2).$$

Taking into consideration that $n_T = \dim_{\mathbb{C}} \mathcal{R} = 2$ we compute the above integrals and get

$$(4.34) \quad \nu(C(\mathcal{H}^\varepsilon(\alpha, \mathcal{C}))) = d_i \cdot \text{Vol} \mathcal{Q}_1(\alpha') \cdot \frac{2\pi\varepsilon^2}{2n_S(2n_S+4)} + o(\varepsilon^2).$$

It remains to note that

$$\nu(C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))) = \dim_{\mathbb{R}} \mathcal{Q}_1(\alpha) \cdot \text{Vol} \mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}),$$

see (4.5), and that

$$\dim_{\mathbb{R}} \mathcal{Q}(\alpha) = 2 \dim_{\mathbb{C}} \mathcal{Q}(\alpha) = 2(\dim_{\mathbb{C}} \mathcal{Q}(\alpha') + 2) = 2n_S + 4.$$

to get

$$(4.35) \quad \text{Vol} \mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}) = \pi\varepsilon^2 \cdot \frac{d_i}{\dim_{\mathbb{C}}(\mathcal{Q}(\alpha)) - 2} \cdot \text{Vol} \mathcal{Q}_1(\alpha') + o(\varepsilon^2).$$

Applying (4.18) and (4.19) to the above expression we obtain (4.28). □

The rest of the discussion in §4.11 also depends on Theorem 1.5. Consider a slightly more general configuration: as before we consider a fixed pair of simple poles P_{j_1}, P_{j_2} , but this time we do not specify which zero do we have at the base of the cylinder. Clearly the corresponding Siegel–Veech constant c_{j_1, j_2}^{pocket} is equal to the sum of the Siegel–Veech constants considered above over all zeroes P_i on our surface S :

$$c_{j_1, j_2}^{pocket} = \sum_{i=1 \mid d_i \geq 1}^k c_i$$

The following Corollary follows immediately from the formula (4.29) above.

Corollary 4.7. *For any stratum of meromorphic quadratic differentials with at most simple poles and with no marked points on \mathbb{CP}^1 and for every fixed pair P_{j_1}, P_{j_2} of simple poles, the Siegel–Veech constant c_{j_1, j_2}^{pocket} is equal to*

$$(4.36) \quad c_{j_1, j_2}^{pocket} = \frac{1}{2\pi^2}$$

Proof. By assumption the stratum $\mathcal{Q}(d_1, \dots, d_k)$ does not contain marked points. We can order d_i in the reverse lexicographic order, so that d_1, \dots, d_m are positive (i.e. correspond to the zeroes) and d_{m+1}, \dots, d_{m+n} are equal to -1 (i.e. correspond to the simple poles).

Since we live on \mathbb{CP}^1 we have $\sum_{i=1}^k d_i = -4$ which is equivalent to $\sum_{i=1}^m d_i = n - 4$. Hence,

$$\begin{aligned} c_{j_1, j_2}^{pocket} &= \frac{1}{2(k-4)} \sum_{i=1}^m (d_i + 1) \frac{1}{\pi^2} = \\ &= \frac{1}{2(n+m-4)} \left(\sum_{i=1}^m d_i + \sum_{i=1}^m 1 \right) \frac{1}{\pi^2} = \frac{1}{2(n+m-4)} (n-4+m) \frac{1}{\pi^2} = \frac{1}{2\pi^2} \end{aligned}$$

□

Proof of Theorem 1.1. Note that Theorem 1.1 counts the number of generalized diagonals joining two fixed corners of a right-angled billiard, while in the “pocket” configuration we count the number of closed flat geodesics on the induced cylinder, which are twice longer. Rescaling, we get an extra factor 4 for the counting problem in this alternative normalization. Applying Theorem 2.4, and taking into consideration the factor $\frac{1}{4}$ in formula (2.1) we get the answer

$$N_{ij}(\Pi, L) \sim \frac{1}{4} \cdot 4 \cdot c_{i,j}^{pocket} \frac{\pi L^2}{\text{Area of the billiard table } \Pi}$$

Plugging in expression (4.36) for $c_{i,j}^{pocket}$ we get formula (1.2). □

4.12. A “dumbbell”, i.e. a simple cylinder separating the sphere and joining a pair of distinct zeroes. .

Consider a configuration \mathcal{C} of type IV, where we have a single cylinder filled with closed regular geodesics, such that the cylinder is bounded by a separatrix loop on each side. We assume that the separatrix loop bounding the cylinder on one side is emitted from a fixed zero P_i of order $d_i \geq 1$ and that the separatrix loop bounding the cylinder on the other side is emitted from a fixed zero P_j of order $d_j \geq 1$.

Such a cylinder separates the original surface S in two parts; let $P_{i_1}, \dots, P_{i_{k_1}}$ be the list of singularities (zeroes and simple poles) which get to the first part and $P_{j_1}, \dots, P_{j_{k_2}}$ be the list of singularities (zeroes and simple poles) which get to the second part. In particular, we have $i \in \{i_1, \dots, i_{k_1}\}$ and $j \in \{j_1, \dots, j_{k_2}\}$. We assume that S does not have any marked points. Denoting as usual by d_n the order of the singularity P_n we can represent the sets with multiplicities $\alpha := \{d_1, \dots, d_k\}$ as a disjoint union of the two subsets

$$\{d_1, \dots, d_k\} = \{d_{i_1}, \dots, d_{i_{k_1}}\} \sqcup \{d_{j_1}, \dots, d_{j_{k_2}}\}.$$

This information is considered to be part of the configuration.

Theorem 4.8. *The Siegel–Veech constant c_C for this configuration is expressed as follows:*

$$(4.37) \quad c_C = \frac{d_i \cdot d_j}{4} \cdot \frac{(\dim_{\mathbb{C}} \mathcal{Q}(d_{i_1}, \dots, d_i - 2, \dots, d_{i_{k_1}}) - 1)! (\dim_{\mathbb{C}} \mathcal{Q}(d_{j_1}, \dots, d_j - 2, \dots, d_{j_{k_2}}) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(d_1, d_2, \dots, d_k) - 2)!} \cdot \frac{\text{Vol } \mathcal{Q}_1(d_{i_1}, \dots, d_i - 2, \dots, d_{i_{k_1}}) \cdot \text{Vol } \mathcal{Q}_1(d_{j_1}, \dots, d_j - 2, \dots, d_{j_{k_2}})}{\text{Vol } \mathcal{Q}_1(d_1, \dots, d_k)}.$$

Plugging in Theorem 1.5 we get:

$$(4.38) \quad c_C = \frac{(d_i + 1)(d_j + 1)}{2} \cdot \frac{(k_1 - 3)! (k_2 - 3)!}{(k - 4)!} \cdot \frac{1}{\pi^2}$$

Proof of (4.37). The proof is completely analogous to computation of the Siegel–Veech constant for configuration III. Denote by α'_a the set with multiplicities obtained from $\{d_{i_1}, \dots, d_{i_{k_1}}\}$ by replacing the entry d_i by $d_i - 2$. Similarly, denote by α'_b the set with multiplicities obtained from $\{d_{j_1}, \dots, d_{j_{k_2}}\}$ by replacing the entry d_j by $d_j - 2$. Define $\alpha' := \alpha'_a \sqcup \alpha'_b$. Contracting the two saddle connections we get a disconnected flat surface S' in the principal boundary stratum $\mathcal{Q}(\alpha')$.

Given a flat surface $S' \in \mathcal{Q}(\alpha')$ and a holonomy vector $\pm \vec{v}$ we have d_i separatrix rays in direction $\pm \vec{v}$ adjacent to the point P_i of S' and d_j separatrix rays in direction $\pm \vec{v}$ adjacent to the point P_j .

Following line-by-line the proof of (4.28) in the previous section we get an expression for $\text{Vol } \mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C})$ completely analogous to (4.35): the only adjustment consists in replacing the factor d_i by the product $d_i d_j$:

$$\text{Vol } \mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}) = \pi \varepsilon^2 \cdot \frac{d_i d_j}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2} \cdot \text{Vol } \mathcal{Q}_1(\alpha') + o(\varepsilon^2)$$

Applying expression (4.9) from §4.4 for $\text{Vol } \mathcal{Q}_1(\alpha')$ and taking into consideration that $\dim_{\mathbb{C}} \mathcal{Q}(\alpha') = \dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2$ we can rewrite the latter expression as

$$\begin{aligned} \text{Vol } \mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}) &= \pi \varepsilon^2 \cdot \frac{d_i d_j}{2} \cdot \frac{(\dim_{\mathbb{C}} \mathcal{Q}(d_{i_1}, \dots, d_i - 2, \dots, d_{i_{k_1}}) - 1)! (\dim_{\mathbb{C}} \mathcal{Q}(d_{j_1}, \dots, d_j - 2, \dots, d_{j_{k_2}}) - 1)!}{(\dim_{\mathbb{C}} \mathcal{Q}(d_1, d_2, \dots, d_k) - 2)!} \cdot \\ &\quad \text{Vol } \mathcal{Q}_1(d_{i_1}, \dots, d_i - 2, \dots, d_{i_{k_1}}) \cdot \text{Vol } \mathcal{Q}_1(d_{j_1}, \dots, d_j - 2, \dots, d_{j_{k_2}}) + o(\varepsilon^2) \end{aligned}$$

Applying (4.18) and (4.19) to the above expression we obtain (4.37). \square

4.13. Siegel–Veech constant c_{area} . Consider an $\text{SL}(2, \mathbb{R})$ -invariant manifold in a stratum of Abelian differentials or a $\text{PSL}(2, \mathbb{R})$ -invariant manifold in a stratum of quadratic differentials. Denote by c_{cyl} the associated Siegel–Veech constant responsible for counting the maximal cylinders of closed geodesics and denote by c_{area} the Siegel–Veech constant responsible for counting the cylinders of closed geodesics counted with weight

$$\frac{(\text{area of the cylinder})}{(\text{area of the surface})}.$$

In [Vo05] Ya. Vorobets proved the following result:

Theorem (Vorobets, 2003). *For any connected component $\mathcal{H}^{\text{comp}}(\alpha)$ of any stratum of Abelian differentials and for almost any flat surface $S \in \mathcal{H}_1^{\text{comp}}(\alpha)$ the ratio of Siegel–Veech constants $c_{\text{area}}/c_{\text{cyl}}$ satisfies the following relation:*

$$\frac{c_{\text{area}}}{c_{\text{cyl}}} = \frac{1}{2g - 2 + n} = \frac{1}{\dim_{\mathbb{C}} \mathcal{H}(\alpha) - 1}.$$

Note that a configuration of $\hat{\text{h}}$ omologous saddle connections of \mathbb{CP}^1 involves at most one cylinder. The following proposition states the Vorobets formula for individual configurations involving cylinders for strata of meromorphic quadratic differentials with simple poles on \mathbb{CP}^1 .

Proposition 4.9. *For any stratum $\mathcal{Q}_1(d_1, \dots, d_n)$ of meromorphic quadratic differentials with simple poles on \mathbb{CP}^1 and for any admissible configuration \mathcal{C} of saddle connections involving a cylinder the following equality holds:*

$$\frac{c_{\text{area}}(\mathcal{C})}{c(\mathcal{C})} = \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_n) - 1} = \frac{1}{n - 3}.$$

Proof. The proof consists in an elementary adjustment of the computation from the previous two sections. We will present the computation of $c_{\text{area}}(\mathcal{C})$ for the “pocket configuration” (configuration of type III) following the analogous computation in §4.11. The computation for the configuration of type IV is completely analogous and is omitted.

This time we have to compute the integral of the ratio $\frac{r_T^2}{r_S^2 + r_T^2}$ of the area $r_T^2/2$ of the cylinder over the total area $(r_S^2 + r_T^2)/2$ of the entire surface. We integrate this expression over $\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C})$. Note that this ratio is the same for proportional surfaces. Thus we can integrate with respect to the corresponding cone $C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))$:

$$\int_{\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C})} \frac{r_T^2}{r_S^2 + r_T^2} d\mu_1 = \dim_{\mathbb{R}} \mathcal{Q}(\alpha) \cdot \int_{C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))} \frac{r_T^2}{r_S^2 + r_T^2} d\nu(S)$$

Moreover, the ratio of the corresponding Siegel–Veech constants satisfies

$$(4.39) \quad \frac{c_{\text{area}}(\alpha, \mathcal{C})}{c_{\text{cyl}}(\alpha, \mathcal{C})} = \lim_{\varepsilon \rightarrow 0} \frac{\int_{C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))} \frac{r_T^2}{r_S^2 + r_T^2} d\nu(S)}{\int_{C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))} d\nu(S)}.$$

The denominator

$$\int_{C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))} d\nu(S) = \nu(C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C})))$$

of the above ratio is given by (4.33). To evaluate the integral in the numerator we modify (4.33) by multiplying the function inside the integral by an extra factor $r_T^2/(r_S^2 + r_T^2)$ obtaining:

$$(4.40) \quad \int_{C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))} \frac{r_T^2}{r_S^2 + r_T^2} d\nu(S) = d_i \cdot \text{Vol}(\mathcal{Q}_1(\alpha')) \cdot \\ \cdot 2\pi\varepsilon^2 \cdot \int_0^1 r_S^{2n_S-1} dr_S \int_0^{\sqrt{1-r_S^2}} r_T^{2n_T-1} dr_T + o(\varepsilon^2),$$

Taking into consideration that $n_T = \dim_{\mathbb{C}} \mathcal{R} = 2$ and evaluating the latter integral we obtain

$$(4.41) \quad \int_{C(\mathcal{Q}_1^\varepsilon(\alpha, \mathcal{C}))} \frac{r_T^2}{r_S^2 + r_T^2} d\nu(S) = d_i \cdot \text{Vol } \mathcal{Q}_1(\alpha') \cdot \frac{4\pi\varepsilon^2}{2n_S(2n_S+2)(2n_S+4)} + o(\varepsilon^2).$$

Plugging (4.41) and (4.34) into expression (4.39) and recalling the definition

$$n_S = \dim_{\mathbb{C}} \mathcal{Q}(\alpha') = \dim_{\mathbb{C}} \mathcal{Q}(\alpha') = \dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 2$$

we obtain

$$\frac{c_{area}(\alpha, \mathcal{C})}{c_{cyl}(\alpha, \mathcal{C})} = \frac{2}{2n_S+2} = \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(\alpha) - 1},$$

which completes the proof of proposition 4.9. \square

Proposition 4.9 immediately implies the following statement.

Corollary 4.10. *For any stratum $\mathcal{Q}_1(d_1, \dots, d_n)$ of meromorphic quadratic differentials with simple poles on \mathbb{CP}^1 the Siegel–Veech constant c_{area} is expressed in terms of the Siegel–Veech constants of configurations as follows:*

$$c_{area} = \frac{1}{n-3} \cdot \sum_{\substack{\text{Configurations } \mathcal{C} \\ \text{containing a cylinder}}} c_{\mathcal{C}}.$$

5. COMPUTATION OF THE VOLUMES OF THE MODULI SPACES

In this section, we prove Theorem 1.5. The approach taken here is somewhat indirect.

5.1. An identity for the Siegel–Veech constant. The idea is to prove the formula Theorem 1.5 for the volume by induction, using the formulas expressing Siegel–Veech constants in terms of the volumes. Namely, by [EKZ, Theorem 3] one has:

$$c_{area}(\mathcal{Q}(d_1, \dots, d_k)) = -\frac{1}{8\pi^2} \sum_{j=1}^k \frac{d_j(d_j+4)}{d_j+2}.$$

On the other hand, by the Vorobets formula applied to \mathbb{CP}^1 (see Corollary 4.10 in §4.13) one has

$$c_{area}(\mathcal{Q}(d_1, \dots, d_k)) = \frac{1}{\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k) - 1} \cdot \sum_{\substack{\text{Configurations } \mathcal{C} \\ \text{containing a cylinder}}} c_{\mathcal{C}}.$$

In view of §2.1, for \mathbb{CP}^1 there are exactly two configurations containing a cylinder: a “pocket” and a “dumbbell”. The formulas for the Siegel–Veech constants were given in §4.

Taking into consideration that $\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k) - 1 = k - 3$, for any collection d_1, \dots, d_k of integers in $\{-1, 1, 2, 3\}$ satisfying the relation

$$\sum_{i=1}^k d_i = -4$$

we get the following identity:

$$(5.1) \quad -\frac{1}{8\pi^2} \sum_{j=1}^k \frac{d_j(d_j+4)}{d_j+2} = \frac{1}{k-3} \cdot \left(\sum_{\substack{\text{“pocket”} \\ \text{configurations}}} c_C + \sum_{\substack{\text{“dumbbell”} \\ \text{configurations}}} c_C \right).$$

If we plug in the expressions (4.28) and (4.37) into (5.1), we get a formula of the form:

$$(5.2) \quad \text{Vol } \mathcal{Q}_1(d_1, \dots, d_n) = \text{Explicit polynomial in volumes of simpler strata.}$$

The formulas (5.2) clearly determine the volumes. Thus, to prove Theorem 1.5, it is enough to show that the expressions for the volumes given by Theorem 1.5 satisfy the recurrence relation (5.2), or equivalently to prove the following:

Theorem 5.1. *The explicit expressions (4.29) and (4.38) for the Siegel–Veech constants satisfy the identity (5.1).*

The proof of the Theorem 5.1 is quite involved and is done in Appendix A. This completes the proof of Theorem 1.5.

6. COUNTING TRAJECTORIES AND ERGODIC THEORY ON MODULI SPACE

In this section we will prove Theorem 2.4 and Theorem 2.6. We modify appropriately the strategy from §4.5 to obtain the weak asymptotic formula (2.1) and the average formula (2.2).

6.1. Pointwise weak asymptotics. For $q \in \mathcal{Q}_1$, define

$$(6.1) \quad M(q, t) = \frac{1}{t} \int_0^t N_C(q, e^s) e^{-2s} ds.$$

A similar calculation to (4.15) shows that, for f the indicator function of the trapezoid \mathcal{T} ,

$$(6.2) \quad M(q, t) \approx \frac{4}{3}\pi \left(\frac{1}{t} \int_0^t \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}(g_s r_\theta q) d\theta ds \right).$$

To study the convergence of the integral

$$(6.3) \quad \frac{1}{t} \int_0^t \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}(g_s r_\theta q) d\theta ds$$

we switch the order of integration, to obtain

$$\frac{1}{2\pi} \int_0^{2\pi} \left(\frac{1}{t} \int_0^t \widehat{f}(g_t r_\theta q) dt \right) d\theta.$$

Let

$$\bar{f}_{\Pi,t}(\theta) := \frac{1}{t} \int_0^t \widehat{f}(g_s r_\theta q_\Pi) dt.$$

By Corollary 3.3, we know that for almost every $\Pi \in \mathcal{B}$, and almost every $\theta \in [0, 2\pi)$, we have

$$\lim_{t \rightarrow \infty} \bar{f}_{\Pi,t}(\theta) = \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \widehat{f}(q) d\mu_1(q) = \frac{3}{4} b_c(\mathcal{Q}).$$

Let us fix Π to be in this full measure set. To prove the convergence of

$$\int_0^{2\pi} \left(\frac{1}{t} \int_0^t \widehat{f}(g_t r_\theta q_\Pi) dt \right) d\theta = \int_0^{2\pi} \bar{f}_{\Pi,t}(\theta) d\theta$$

we follow an argument from [EMS03]. Note that if \widehat{f} were continuous and compactly supported, the dominated convergence theorem would yield the result. We will approximate \widehat{f} by such functions.

Fix $\varepsilon > 0$, let $h_\varepsilon : \mathcal{Q} \rightarrow \mathbb{R}$ be a continuous function with

$$(6.4) \quad h_\varepsilon(q) = \begin{cases} 1 & l(q) > \varepsilon \\ 0 & l(q) < \varepsilon/2 \end{cases}$$

Here, $l(q)$ denotes the length of the shortest saddle connection on q . The function h_ε is a smoothed version of the indicator function of the compact part of the stratum \mathcal{Q} .

We introduce some more notation. Given $\phi : \mathcal{Q}_1 \rightarrow \mathbb{R}$, define

$$(6.5) \quad (A_t \phi)(q) = \frac{1}{2\pi} \int_0^{2\pi} \phi(g_t r_\theta q) d\theta.$$

With this notation, our integral (6.3) is given by

$$\frac{1}{t} \int_0^t (A_s \widehat{f})(q_\Pi) ds.$$

For any $q \in \mathcal{Q}_1$,

$$(6.6) \quad (A_t(\widehat{f}h_\varepsilon))(q) \leq (A_t \widehat{f})(q) = (A_t(\widehat{f}h_\varepsilon))(q) + (A_t(\widehat{f}(1-h_\varepsilon)))(q).$$

We follow [EMS03, p.435, proof of Theorem 2.4]. Fix $1 > \eta > \delta > 0$. [EM00, Theorem 5.1] shows there is a $C(\delta)$ so that for all $q \in \mathcal{Q}$

$$(6.7) \quad \widehat{f}(q) \leq \frac{C(\delta)}{l(q)^{1+\delta}}.$$

On the other hand, $1 - h_\varepsilon(q) > 0$ implies $l(q) \leq \varepsilon$, so

$$\widehat{f}(q)(1 - h_\varepsilon(q)) \leq \widehat{f}(q) \leq \frac{C(\delta)}{l(q)^{1+\eta}} \cdot l(q)^{\eta-\delta} \leq \varepsilon^{\eta-\delta} \frac{C(\delta)}{l(q)^{1+\eta}}.$$

Thus,

$$(A_t(\widehat{f}(1-h_\varepsilon)))(q) \leq C(\delta) \varepsilon^{\eta-\delta} (A_t l^{-1-\eta})(q).$$

[EM00, Theorem 5.2] states that for $\eta < 1$, there is a $C_1 = C_1(\eta, \Pi)$ so that for all $t > 0$,

$$(A_t l^{-1-\eta})(q_\Pi) < C_1(\eta, \Pi)$$

Since $\widehat{f}h_\varepsilon$ is continuous and compactly supported, for any Π from the set of full measure to which Corollary 3.3 applies we get

$$\lim_{t \rightarrow \infty} \int_0^t \left(A_s(\widehat{f}h_\varepsilon) \right) (q_\Pi) ds = \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \widehat{f}h_\varepsilon(q) d\mu_1(q)$$

So we have

$$(6.8) \quad \liminf_{t \rightarrow \infty} \int_0^t \left(A_s \widehat{f} \right) (q_\Pi) ds \geq \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \widehat{f}h_\varepsilon(q) d\mu_1(q)$$

and

$$(6.9) \quad \limsup_{t \rightarrow \infty} \frac{1}{t} \int_0^t A_s(\widehat{f}) ds \leq \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} \widehat{f}h_\varepsilon(q) d\mu_1(q) + C(\delta)C_1(\eta, \Pi)\varepsilon^{\eta-\delta}.$$

Combining (6.8) and (6.9) and letting $\varepsilon \rightarrow 0$, we obtain our desired result. Theorem 2.4 is proved. \square

We complete this section with the proof of Theorem 1.4 from §1.1.

Proof of Theorem 1.4. The proof is completely analogous to the proof of Theorem 2.4 above; we just have to carefully follow the normalization which is different from the previous case.

By [EKZ, Theorem 3] one has:

$$(6.10) \quad c_{area}(\mathcal{Q}(k_1 - 2, \dots, k_n - 2)) = -\frac{1}{8\pi^2} \sum_{j=1}^n \frac{(k_j - 2)(k_j + 2)}{k_j} = \frac{1}{8\pi^2} \sum_{j=1}^n \left(\frac{4}{k_j} - k_j \right).$$

The length of a closed trajectory in Π is the same as the length of the associated closed geodesic on the covering flat sphere S . By definition, the area of the band of closed trajectories on Π is the same as the area of each of the maximal cylinders on \mathbb{CP}^1 . However, the flat area of S is twice the area of Π . Thus, the ratio

$$\frac{(\text{area of the band of periodic trajectories on } \Pi)}{(\text{area of } \Pi)}$$

is twice larger then the corresponding ratio

$$\frac{(\text{area of the maximal cylinder of periodic geodesics on } S)}{(\text{area of } S)}.$$

Taking into consideration that the bands of closed trajectories in the polygon Π are in the natural one-to-two correspondence with the maximal cylinders of closed regular geodesics on the covering flat sphere S , see Figure 3, we get

$$N_{area}(\Pi, L) = N_{area}(S, L).$$

It remains to note that

$$N_{area}(S, L) \sim c_{area}(\mathcal{Q}) \cdot \frac{L^2}{\text{area}(S)} = \frac{c_{area}(\mathcal{Q})}{2} \cdot \frac{L^2}{\text{area}(\Pi)}$$

to conclude that the constant in the weak quadratic asymptotic (1.4) in Theorem 1.4 is one half of the Siegel–Veech constant $c_{area}(\mathcal{Q})$ from (6.10). \square

6.2. L^1 results. Here, we use (3.3) from Corollary 3.3. Again letting f be the indicator function of the trapezoid \mathcal{T} , defined above, and using the reduction (4.15), we can reduce studying

$$\lim_{t \rightarrow \infty} \int_A e^{-2t} N_{\mathcal{C}}(\Pi, e^t) d\mu_{\mathcal{B}}(\Pi)$$

to studying

$$\lim_{t \rightarrow \infty} \frac{4}{3} \pi \frac{1}{\mu_{\mathcal{B}}(A)} \int_A \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}(g_T r_{\theta} q_{\Pi}) d\mu_{\mathcal{B}}(\Pi),$$

where \widehat{f} is as in (4.12). As in §6.1, we approximate \widehat{f} by the compactly supported function $\widehat{f}h_{\varepsilon}$, and applying the equidistribution result (3.3), we have

$$\lim_{t \rightarrow \infty} \frac{4}{3} \pi \frac{1}{\mu_{\mathcal{B}}(A)} \int_A \frac{1}{2\pi} \int_0^{2\pi} \widehat{f}(g_T r_{\theta} q_{\Pi}) d\mu_{\mathcal{B}}(\Pi) = \frac{4}{3} \pi \frac{1}{\mu_1(\mathcal{Q}_1)} \int_{\mathcal{Q}_1} f(q) d\mu_1(q) = \pi b_{\mathcal{C}}(\mathcal{Q}).$$

Theorem 2.6 is proved.

A. PROOF OF COMBINATORIAL IDENTITY

In this appendix, we prove Theorem 5.1.

A.1. General identity to prove. Let d_1, \dots, d_m be the degrees of zeroes only. Let the number n of simple poles is expressed as $n = 4 + \sum_{i=1}^m d_i$. The total number $k = m + n$ of all singularities is, thus, expressed as $k = 4 + \sum_{i=1}^m (d_i + 1)$.

For each choice of the zero in the “pocket” configuration there are

$$\frac{n}{2} = \frac{4 + \sum_{i=1}^m (d_i + 1)}{2}$$

ways to chose a pair of distinguished poles. For each partition of the set $\{1, \dots, m\}$ enumerating the zeroes into two nonempty subsets $\{i_1, \dots, i_{m_1}\} \sqcup \{j_1, \dots, j_{m_2}\}$ (which makes part of the “dumbbell” configuration) there are

$$\frac{(4 + \sum_{i=1}^m d_i)!}{(2 + \sum_{i=1}^{m_1} d_i)!(2 + \sum_{j=1}^{m_2} d_j)!}$$

ways to partition the simple poles between two parts of the “dumbbell”. Taking into consideration this counting and plugging in the explicit conjectural expressions (4.29) and (4.38) for the Siegel–Veech constants $c_{\mathcal{C}}$ into (5.1) we observe that

the right-hand-side of (5.1) can be read as

$$\begin{aligned}
& \frac{1}{k-3} \cdot \left(\sum_{\substack{\text{"pocket"} \\ \text{configurations}}} c_{\mathcal{C}} + \sum_{\substack{\text{"dumbbell"} \\ \text{configurations}}} c_{\mathcal{C}} \right) \stackrel{?}{=} \\
& \stackrel{?}{=} \frac{1}{2\pi^2} \cdot \frac{1}{(1 + \sum_{i=1}^m (d_i + 1))} \cdot \left(\binom{4 + \sum_{i=1}^m d_i}{2} + \right. \\
& + \sum_{1 \leq i < j \leq m} (d_i + 1)(d_j + 1) \sum_{\substack{\text{partitions of } \{1, \dots, m\} \text{ into} \\ \{i_1, \dots, i_{m_1}\} \sqcup \{j_1, \dots, j_{m_2}\} \\ \text{such that } i \text{ is in the first subset} \\ \text{and } j \text{ is in the second subset}}} \frac{(4 + \sum_{i=1}^m d_i)!}{(2 + \sum_{i=1}^{m_1} d_i)!(2 + \sum_{j=1}^{m_2} d_j)!} \cdot \\
& \left. \frac{(2 - 3 + \sum_{i=1}^{m_1} (d_i + 1))!(2 - 3 + \sum_{j=1}^{m_2} (d_j + 1))!}{(\sum_{i=1}^m (d_i + 1))!} \right)
\end{aligned}$$

Multiplying both parts of the conjectural identity by the common factor

$$4\pi^2 \cdot \left(1 + \sum_{i=1}^m (d_i + 1) \right)$$

moving the binomial coefficient

$$\binom{4 + \sum_{i=1}^m d_i}{2}$$

coming from the “pocket” configuration to the left-hand-side of the identity and simplifying the resulting expressions we get the following conjectural identity:

$$\begin{aligned}
\text{(A.1)} \quad & \left(6 + \sum_{i=1}^m \frac{d_i(d_i + 1)}{d_i + 2} \right) \cdot \left(1 + \sum_{i=1}^m (d_i + 1) \right) - \left(4 + \sum_{i=1}^m d_i \right) \left(3 + \sum_{i=1}^m d_i \right) \stackrel{?}{=} \\
& \stackrel{?}{=} 2 \cdot \frac{(4 + \sum_{i=1}^m d_i)!}{(\sum_{i=1}^m (d_i + 1))!} \cdot \sum_{1 \leq i < j \leq m} (d_i + 1)(d_j + 1) \cdot \\
& \cdot \sum_{\substack{\text{partitions of } \{1, \dots, m\} \text{ into} \\ \{r_1, \dots, r_{m_1}\} \sqcup \{s_1, \dots, s_{m_2}\} \\ \text{such that } i \text{ is in the first subset} \\ \text{and } j \text{ is in the second subset}}} \frac{(-1 + \sum_{i=1}^{m_1} (d_{r_i} + 1))! \cdot (-1 + \sum_{j=1}^{m_2} (d_{s_j} + 1))!}{(2 + \sum_{i=1}^{m_1} d_{r_i})! \cdot (2 + \sum_{j=1}^{m_2} d_{s_j})!}.
\end{aligned}$$

This is the identity which we need to prove.

Changing the order of the summation we can first sum over all possible partitions of the set of indices $\{1, \dots, m\}$ and having chosen the partition we consider all possible ways to select a distinguished element i in the first subset and a distinguished element j in the second subset. Note, however, that we will see each of

the elements of the above sum twice. Thus, collecting the resulting sums we can rewrite the sum in the right-hand-side of the above expression as follows:

$$\begin{aligned}
& \sum_{\substack{\text{partitions of } \{1, \dots, m\} \text{ into} \\ \text{two nonempty sets} \\ \{r_1, \dots, r_{m_1}\} \sqcup \{s_1, \dots, s_{m_2}\}}} \frac{(-1 + \sum_{i=1}^{m_1} (d_{r_i} + 1))! \cdot (-1 + \sum_{j=1}^{m_2} (d_{s_j} + 1))!}{(2 + \sum_{i=1}^{m_1} d_{r_i})! \cdot (2 + \sum_{j=1}^{m_2} d_{s_j})!} \\
& \cdot \left(\sum_{1 \leq i \leq m_1} (d_{r_i} + 1) \right) \left(\sum_{1 \leq j \leq m_2} (d_{s_j} + 1) \right) = \\
& = \sum_{\substack{\text{partitions of } \{1, \dots, m\} \text{ into} \\ \text{two nonempty sets} \\ \{r_1, \dots, r_{m_1}\} \sqcup \{s_1, \dots, s_{m_2}\}}} \frac{(\sum_{i=1}^{m_1} (d_{r_i} + 1))! \cdot (\sum_{j=1}^{m_2} (d_{s_j} + 1))!}{(2 + \sum_{i=1}^{m_1} d_{r_i})! \cdot (2 + \sum_{j=1}^{m_2} d_{s_j})!}.
\end{aligned}$$

Hence, we can rewrite the right-hand-side in (A.1) as a sum over the ratios of binomial coefficients:

$$\begin{aligned}
& \frac{(4 + \sum_{i=1}^m d_i)!}{(\sum_{i=1}^m (d_i + 1))!} \cdot \sum_{\substack{\text{partitions of } \{1, \dots, m\} \text{ into} \\ \text{two nonempty sets} \\ \{r_1, \dots, r_{m_1}\} \sqcup \{s_1, \dots, s_{m_2}\}}} \frac{(\sum_{i=1}^{m_1} (d_{r_i} + 1))! \cdot (\sum_{j=1}^{m_2} (d_{s_j} + 1))!}{(2 + \sum_{i=1}^{m_1} d_{r_i})! \cdot (2 + \sum_{j=1}^{m_2} d_{s_j})!} = \\
& = \sum_{\substack{\text{partitions of } \{1, \dots, m\} \text{ into} \\ \text{two nonempty sets} \\ \{r_1, \dots, r_{m_1}\} \sqcup \{s_1, \dots, s_{m_2}\}}} \frac{\binom{4 + \sum_{i=1}^m d_i}{2 + \sum_{l=1}^{m_1} d_{r_l}}}{\binom{\sum_{i=1}^m (d_i + 1)}{\sum_{l=1}^{m_1} (d_{r_l} + 1)}}.
\end{aligned}$$

Finally, omitting the conditions that the subsets of the partition are nonempty, we get two extra terms. It is immediate to verify that their sum is equal to

$$\left(4 + \sum_{i=1}^m d_i \right) \left(3 + \sum_{i=1}^m d_i \right)$$

and, thus, we can rewrite the needed identity (A.1) as follows:

$$\begin{aligned}
\text{(A.2)} \quad & \left(6 + \sum_{i=1}^m \frac{d_i(d_i + 1)}{d_i + 2} \right) \cdot \left(1 + \sum_{i=1}^m (d_i + 1) \right) \stackrel{?}{=} \\
& \stackrel{?}{=} \sum_{\substack{\text{partitions of } \{1, \dots, m\} \text{ into} \\ \text{two complementary sets} \\ \{r_1, \dots, r_{m_1}\} \sqcup \{s_1, \dots, s_{m_2}\}}} \frac{\binom{4 + \sum_{i=1}^m d_i}{2 + \sum_{l=1}^{m_1} d_{r_l}}}{\binom{\sum_{i=1}^m (d_i + 1)}{\sum_{l=1}^{m_1} (d_{r_l} + 1)}}
\end{aligned}$$

We will show that (A.2) is valid for any nonempty collection of nonnegative integers $\{d_1, \dots, d_m\}$.

A.2. Identity in terms of multinomial coefficients. Let n_d be the total number of entries d in the set (with multiplicities) $\{d_1, \dots, d_m\}$. The left-hand-side of conjectural identity (A.2) can be expressed as

$$\left(6 + \sum_d \frac{d(d+1)}{d+2} n_d\right) \cdot \left(1 + \sum_d (d+1)n_d\right)$$

The right-hand-side can be represented in terms of n_d as

$$\begin{aligned} & \frac{(4 + \sum_d d \cdot n_d)!}{(\sum_d (d+1)n_d)!} \cdot \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \dots \binom{n_1}{k_1} \binom{n_2}{k_2} \dots \\ & \frac{(\sum_d (d+1)k_d)! \cdot (\sum_d (d+1)(n_d - k_d))!}{(2 + \sum_d d \cdot k_d)! \cdot (2 + \sum_d d \cdot (n_d - k_d))!} = \\ & = \frac{n_1! n_2! \dots (4 + \sum_d d \cdot n_d)!}{(\sum_d (d+1)n_d)!} \cdot \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} \dots \dots \dots \\ & \frac{(\sum_d (d+1)k_d)!}{k_1! k_2! \dots (2 + \sum_d d \cdot k_d)!} \cdot \frac{(\sum_d (d+1)(n_d - k_d))!}{(n_1 - k_1)! (n_2 - k_2)! \dots (2 + (\sum_d d \cdot (n_d - k_d)))!} \end{aligned}$$

Note now that the common factor is (up to four missing factors) is a multinomial coefficient and that the bottom line is a product of two “complementary” multinomial coefficients (with two missing factors each).

A.3. Notation. To simplify the otherwise complicated factorials and terms, we introduce some notation: $\mathbf{k} = (k_1, \dots, k_m)$, $\mathbf{s} = (s_1, \dots, s_m)$, and $\mathbf{d} = (d_1, \dots, d_m)$ are all m -tuples. We will think of $\mathbf{k}, \mathbf{d} \in (\mathbb{Z}^{\geq 0})^m$, and \mathbf{s} as variables. We write

$$\mathbf{1} = (1, 1, \dots, 1) := \sum_{i=1}^m \mathbf{e}_i,$$

where \mathbf{e}_i are the standard basis vectors. Let n denote an integer.

Inner Product:

$$\mathbf{k} \cdot \mathbf{d} := \sum_{i=1}^m k_i d_i$$

is the standard inner product.

Factorials:

$$\mathbf{k}! := \prod_{i=1}^m k_i!$$

Multinomial Coefficients:

$$\binom{n}{\mathbf{k}} := \binom{n}{k_1, \dots, k_m, n - \mathbf{k} \cdot \mathbf{1}}$$

Deletion of variables: Here, we can have $i = j$:

$$\mathbf{k}^i = \mathbf{k} - \mathbf{e}_i, \mathbf{k}^{i,j} = \mathbf{k} - \mathbf{e}_i - \mathbf{e}_j$$

Powers:

$$\mathbf{s}^{\mathbf{k}} = \prod_{i=1}^m s_i^{k_i}$$

We redefine notations \mathbf{d} and m denoting from now on by \mathbf{d} the original set $\{d_1, \dots, d_m\}$ with suppressed multiplicities. In other words, we define the new \mathbf{d} as the set of distinct entries of the original set $\{d_1, \dots, d_m\}$. We also redefine m denoting by m the cardinality of the new set \mathbf{d} . Applying manipulations performed in §A.2 we can rewrite the identity we need to prove in the following way:

$$\begin{aligned} & \frac{6 + \sum_{i=1}^m \frac{d_i(d_i+1)}{d_i+2} n_i}{(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{n}) \cdot (3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{n}) \cdot (4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{n})} \cdot \binom{4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{n}}{\mathbf{n}} \stackrel{?}{=} \\ & \stackrel{?}{=} \sum_{\mathbf{k}=0}^{\mathbf{n}} \frac{1}{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) (2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})} \cdot \binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} \cdot \\ & \cdot \frac{1}{(1 + (\mathbf{d} + \mathbf{1}) \cdot (\mathbf{n} - \mathbf{k})) (2 + (\mathbf{d} + \mathbf{1}) \cdot (\mathbf{n} - \mathbf{k}))} \cdot \binom{2 + (\mathbf{d} + \mathbf{1}) \cdot (\mathbf{n} - \mathbf{k})}{\mathbf{n} - \mathbf{k}}. \end{aligned}$$

A.4. Generating Functions. We define

$$F(\mathbf{s}) := \sum_{\mathbf{k} \in (\mathbb{Z}^{\geq 0})^m} \frac{\binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}}}{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) (2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})} \mathbf{s}^{\mathbf{k}},$$

and

$$\begin{aligned} G(\mathbf{s}) := \sum_{\mathbf{k} \in (\mathbb{Z}^{\geq 0})^m} & \frac{6 + \sum_{i=1}^m \frac{d_i(d_i+1)}{d_i+2} k_i}{(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) (3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) (4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})} \cdot \\ & \cdot \binom{4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} \mathbf{s}^{\mathbf{k}}. \end{aligned}$$

In terms of these generating functions, the identity to be proved becomes

$$F^2 \stackrel{?}{=} G.$$

A.5. Mohanty's Formula. Our main tools are the combinatorial identities developed by Mohanty [Mh]. We recall formulas (31) and (32) of [Mh], in our own notation. Given $a \in \mathbb{N}$, $\mathbf{b}, \mathbf{k} \in (\mathbb{Z}^{\geq 0})^m$, define the *Mohanty coefficient*

$$A(a; \mathbf{b}; \mathbf{k}) := \frac{a}{a + \mathbf{b} \cdot \mathbf{k}} \binom{a + \mathbf{b} \cdot \mathbf{k}}{\mathbf{k}}.$$

We have

Theorem A.1. [Mohanty [Mh], (31) and (32)] *With notation as above, we have*

$$\sum_{\mathbf{k} \in (\mathbb{Z}^{\geq 0})^m} A(a; \mathbf{b}; \mathbf{k}) \mathbf{s}^{\mathbf{k}} = z^a,$$

where

$$1 - z + \sum_{i=1}^m s_i z^{b_i} = 0.$$

Since we will use only one \mathbf{b} , namely $\mathbf{b} = \mathbf{d} + \mathbf{1}$, we will abbreviate the Mohanty coefficient by defining $A(a; \mathbf{k}) = A(a; \mathbf{d} + \mathbf{1}; \mathbf{k})$.

In the rest of the appendix we prove:

Theorem A.2.

$$F^2 = G.$$

More precisely,

$$(A.3) \quad G(\mathbf{s}) = F^2(\mathbf{s}) - \frac{1}{4}z^2 \left(1 - z + \sum_{i=1}^m s_i z^{d_i+1} \right)^2,$$

where z is as in Mohanty's formula Theorem A.1 for $A(a; \mathbf{k}) = A(a; \mathbf{d} + \mathbf{1}; \mathbf{k})$, so

$$1 - z + \sum_{i=1}^m s_i z^{d_i+1} = 0.$$

To prove this formula, we will derive formulas for F (§A.6) and G (§A.7), and show (A.3).

A.6. Formula for F . Our first lemma is the formula for F :

Lemma A.3.

$$F(\mathbf{s}) = \sum_{i=1}^m \frac{s_i}{d_i + 2} z^{d_i+2} - \frac{1}{2}z^2 + z,$$

where

$$1 - z + \sum_{i=1}^m s_i z^{d_i+1} = 0.$$

Proof. We expand the right hand side using Mohanty's formula, and equalize the $\mathbf{s}^{\mathbf{k}}$ terms of the right hand and left hand sides. The right hand side expands, term-by-term, as:

$$\begin{aligned} \sum_{i=1}^m \frac{s_i}{d_i + 2} z^{d_i+2} &\mapsto \sum_{i=1}^m \frac{s_i}{d_i + 2} \frac{d_i + 2}{d_i + 2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{d_i + 2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} \\ \frac{1}{2}z^2 &\mapsto \frac{1}{2} \frac{2}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} \\ z &\mapsto \frac{1}{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} \end{aligned}$$

The $\mathbf{s}^{\mathbf{k}}$ terms of each of the second and third expressions can be read off directly. For the first, we have:

$$\sum_{i=1}^m \frac{s_i}{d_i + 2} z^{d_i+2} \mapsto \sum_{i=1}^m \frac{1}{d_i + 2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^i} \binom{d_i + 2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^i}{\mathbf{k}^i}$$

Observing that

$$a + d_i + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^i = a - 1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k},$$

we can re-write this as

$$\sum_{i=1}^m \frac{s_i}{d_i + 2} z^{d_i+2} \mapsto \sum_{i=1}^m \frac{1}{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}^i}$$

Thus, our identity reduces to showing that :

$$\frac{\binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}}}{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})}$$

is the sum of

$$\sum_{i=1}^m \frac{1}{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}^i},$$

and

$$\frac{1}{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} - \frac{1}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}}$$

Multiplying through by

$$(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}),$$

our identity reduces to showing that

$$\binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}}$$

equals

$$(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) \left(\binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} + \sum_{i=1}^m \binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}^i} \right) - (1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) \binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}}$$

Moving the last term to the left hand side, and canceling the resulting $(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})$, our identity reduces to:

$$\binom{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} = \binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} + \sum_{i=1}^m \binom{1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}^i},$$

which is the basic identity for multinomial coefficients

$$\binom{n}{\mathbf{k}} = \binom{n-1}{\mathbf{k}} + \sum_{i=1}^m \binom{n-1}{\mathbf{k}^i},$$

with $n = 2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}$. □

A.7. Formula for G . Our second main lemma is a formula for G :

Lemma A.4.

$$G(\mathbf{s}) = \frac{3}{4}z^2 - \frac{1}{2}z^3 + \frac{1}{2} \left(\sum_{i=1}^m \frac{d_i s_i z^{d_i+4}}{d_i + 2} - \sum_{i=1}^m \frac{(d_i - 2) s_i z^{d_i+3}}{d_i + 2} - \sum_{i,j=1}^m \frac{d_i (d_i + 4) s_i s_j z^{4+d_i+d_j}}{(d_i + 2)(4 + d_i + d_j)} \right).$$

Before we prove this lemma, we prove Theorem A.2 assuming it.

A.8. Proof of Theorem A.2. We want to show (A.3) assuming Lemma A.4 (and Lemma A.3), that is, we want to show

$$(A.4) \quad G(\mathbf{s}) \stackrel{?}{=} F^2(\mathbf{s}) - \frac{1}{4}z^2 \left(1 - z + \sum_{i=1}^m s_i z^{d_i+1} \right)^2.$$

Expanding $F^2(\mathbf{s})$ using

$$F(\mathbf{s}) = \sum_{i=1}^m \frac{s_i}{d_i+2} z^{d_i+2} - \frac{1}{2}z^2 + z,$$

we obtain

$$F^2(\mathbf{s}) = \left(\sum_{i=1}^m \frac{s_i}{d_i+2} z^{d_i+2} \right)^2 + \left(z - \frac{1}{2}z^2 \right)^2 + 2 \left(\sum_{i=1}^m \frac{s_i}{d_i+2} z^{d_i+2} \right) \left(z - \frac{1}{2}z^2 \right)$$

Expanding this expression for $F^2(\mathbf{s})$, expanding the second term in the right-hand side of (A.4) and simplifying, we obtain three types of terms in the resulting expression for the right-hand side of (A.4):

Simple powers: $\frac{3}{4}z^2 - \frac{1}{2}z^3$

Single sums: $\sum_{i=1}^m \left(\frac{2}{d_i+2} - \frac{1}{2} \right) s_i z^{d_i+3} + \sum_{i=1}^m \left(\frac{1}{2} - \frac{1}{d_i+2} \right) s_i z^{d_i+4}$

Double sum: $\sum_{i,j=1}^m \left(\frac{1}{(d_i+2)(d_j+2)} - \frac{1}{4} \right) s_i s_j z^{4+d_i+d_j}$

Expanding Lemma A.4 in a similar fashion, we have the corresponding terms for $G(\mathbf{s})$:

Simple powers: $\frac{3}{4}z^2 - \frac{1}{2}z^3$

Single sums: $\sum_{i=1}^m \left(\frac{2-d_i}{2(d_i+2)} \right) s_i z^{d_i+3} + \sum_{i=1}^m \left(\frac{d_i}{2(d_i+2)} \right) s_i z^{d_i+4}$

Double sum: $-\sum_{i,j=1}^m \left(\frac{d_i(d_i+4)}{2(d_i+2)} \frac{1}{4+d_i+d_j} \right) s_i s_j z^{4+d_i+d_j}$

A quick inspection shows that the simple powers and single sums are equal. For the double sum, we need to combine the (i, j) and (j, i) terms in both sums (note that the terms are identical in the F^2 expansion, but not in the G expansion), and check their equality. The F^2 term is thus

$$-2 \left(\frac{1}{4} - \frac{1}{(d_i+2)(d_j+2)} \right)$$

and the G term is

$$-\frac{1}{4+d_i+d_j} \left(\frac{d_i(d_i+4)}{2(d_i+2)} + \frac{d_j(d_j+4)}{2(d_j+2)} \right).$$

To check their equality, we reorganize and obtain:

$$\frac{-4 + (d_i+2)(d_j+2)}{2(d_i+2)(d_j+2)} \stackrel{?}{=} \frac{1}{2(4+d_i+d_j)} \left(\frac{d_i(d_i+4)(d_j+2) + d_j(d_j+4)(d_i+2)}{(d_i+2)(d_j+2)} \right)$$

Cancelling and cross-multiplying, this reduces to

$$(4+d_i+d_j)(d_i d_j + 2d_i + 2d_j) = (d_i^2 + 4d_i)(d_j+2) + (d_j^2 + 4d_j)(d_i+2),$$

which is easily verified. \square

A.9. Proof of Lemma A.4. Recall that the $\mathbf{s}^{\mathbf{k}}$ term for G is

$$(A.5) \quad \frac{6 + \sum_{i=1}^m \frac{d_i(d_i+1)}{d_i+2} k_i}{(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})(4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})} \binom{4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}}$$

We observe

$$\frac{d_i(d_i+1)}{(d_i+2)} = d_i - 1 + \frac{2}{d_i+2},$$

and write

$$6 = \frac{3}{2} ((4 + \mathbf{d} \cdot \mathbf{k}) - \mathbf{d} \cdot \mathbf{k}).$$

Using these, we rewrite the term (A.5) as the product of three terms:

$$\textbf{Numerator: } \left(\frac{3}{2}(4 + \mathbf{d} \cdot \mathbf{k}) - \sum_{i=1}^m \left(\frac{1}{2}d_i + 1 - \frac{2}{d_i+2} \right) k_i \right)$$

$$\textbf{Partial Fractions: } \left(\frac{1}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} - \frac{1}{3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \right)$$

$$\textbf{Multinomial Coefficient: } \frac{1}{4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \binom{4 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}}{\mathbf{k}} = \frac{(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!}$$

We consider terms from this triple product in turn.

A.9.1. k_i -terms. First, we consider the individual term

$$\left(\frac{1}{2}d_i + 1 - \frac{2}{d_i+2} \right) k_i \left(\frac{1}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} - \frac{1}{3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \right) \frac{(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!}.$$

Keeping the $\left(\frac{1}{2}d_i + 1 - \frac{2}{d_i+2} \right)$ term outside for now, and considering only the first part of the difference, we are interested in

$$k_i \frac{1}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \frac{(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!} = (3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) \frac{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}^i!(4 + \mathbf{d} \cdot \mathbf{k})!}$$

Expanding

$$(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) = (4 + \mathbf{d} \cdot \mathbf{k}) + \mathbf{1} \cdot \mathbf{k}^i,$$

we first consider

$$\begin{aligned} (4 + \mathbf{d} \cdot \mathbf{k}) \frac{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}^i!(4 + \mathbf{d} \cdot \mathbf{k})!} &= \frac{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}^i!(3 + \mathbf{d} \cdot \mathbf{k})!} \\ &= \frac{(d_i + 2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^i)!}{\mathbf{k}^i!(d_i + 3 + \mathbf{d} \cdot \mathbf{k}^i)!} \\ &= \frac{1}{d_i + 3} \frac{d_i + 3}{d_i + 3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^i} \binom{d_i + 3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^i}{\mathbf{k}^i} \\ &= \frac{1}{d_i + 3} A(d_i + 3; \mathbf{k}^i) \end{aligned}$$

Now expanding $\mathbf{1} \cdot \mathbf{k}^i = (k_i - 1) + \sum_{j \neq i} k_j$, we have the terms

$$\begin{aligned} (k_i - 1) \frac{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}^i!(4 + \mathbf{d} \cdot \mathbf{k})!} &= \frac{1}{4 + 2d_i} \frac{4 + 2d_i}{4 + 2d_i + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^{i,i}} \binom{4 + 2d_i + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^{i,i}}{\mathbf{k}^{i,i}} \\ &= \frac{1}{4 + 2d_i} A(4 + 2d_i; \mathbf{k}^{i,i}) \end{aligned}$$

and

$$\begin{aligned} (k_j) \frac{(1 + (\mathbf{d} + \mathbf{1})\mathbf{k})!}{\mathbf{k}^i!(4 + \mathbf{d} \cdot \mathbf{k})!} &= \frac{1}{4 + d_i + d_j} \frac{4 + d_i + d_j}{4 + d_i + d_j + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^{i,j}} \binom{4 + d_i + d_j + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^{i,j}}{\mathbf{k}^{i,j}} \\ &= \frac{1}{4 + d_i + d_j} A(4 + d_i + d_j; \mathbf{k}^{i,j}) \end{aligned}$$

Collecting all of these, we have

$$\begin{aligned} k_i \frac{1}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \frac{(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!} &= \\ &= \frac{1}{d_i + 3} A(d_i + 3; \mathbf{k}^i) + \sum_{j=1}^m \frac{1}{4 + d_i + d_j} A(4 + d_i + d_j; \mathbf{k}^{i,j}) \end{aligned}$$

Next, we work with the factor

$$\begin{aligned} -k_i \frac{1}{3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \frac{(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!} &= -\frac{(2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}^i!(4 + \mathbf{d} \cdot \mathbf{k})!} \\ &= \frac{(3 + d_i + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}^i)!}{\mathbf{k}^i!(4 + d_i + \mathbf{d} \cdot \mathbf{k}^i)!} \\ &= -\frac{1}{d_i + 4} A(4 + d_i; \mathbf{k}^i) \end{aligned}$$

These k_i terms come with the factor of

$$\left(\frac{1}{2} d_i + 1 - \frac{2}{d_i + 2} \right) = \frac{d_i(d_i + 4)}{2(d_i + 2)},$$

so we have that their total contribution is:

$$\sum_{i=1}^m \frac{d_i(d_i + 4)}{2(d_i + 2)} \left(\frac{A(d_i + 3; \mathbf{k}^i)}{d_i + 3} - \frac{A(d_i + 4; \mathbf{k}^i)}{d_i + 4} + \sum_{j=1}^m \frac{A(4 + d_i + d_j; \mathbf{k}^{i,j})}{4 + d_i + d_j} \right).$$

Summing over $\mathbf{k} \in (\mathbb{Z}^{\geq 0})^m$, we obtain, using Mohanty's formula,

$$(A.6) \quad \sum_{i=1}^m \frac{d_i(d_i + 4)}{2(d_i + 2)} \left(\frac{s_i z^{d_i+3}}{d_i + 3} - \frac{s_i z^{d_i+4}}{d_i + 4} + \sum_{j=1}^m \frac{s_i s_j z^{4+d_i+d_j}}{4 + d_i + d_j} \right)$$

A.9.2. $\frac{3}{2}(4 + \mathbf{d} \cdot \mathbf{k})$ -terms. We now expand the $\frac{3}{2}(4 + \mathbf{d} \cdot \mathbf{k})$ -terms, keeping $\frac{3}{2}$ on the outside for now. That is, we consider

$$(4 + \mathbf{d} \cdot \mathbf{k}) \left(\frac{1}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} - \frac{1}{3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \right) \frac{(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!}.$$

As above, we first work with the term

$$\begin{aligned} (4 + \mathbf{d} \cdot \mathbf{k}) \frac{1}{2 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}} \frac{(3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!} &= (3 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k}) \frac{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(3 + \mathbf{d} \cdot \mathbf{k})!} \\ &= ((3 + \mathbf{d} \cdot \mathbf{k}) + \mathbf{1} \cdot \mathbf{k}) \frac{(1 + (\mathbf{d} + \mathbf{1}) \cdot \mathbf{k})!}{\mathbf{k}!(3 + \mathbf{d} \cdot \mathbf{k})!} \end{aligned}$$

The $\mathbf{1} \cdot \mathbf{k}$ term can be split up into individual terms, and as above, we have

$$k_i \frac{(1 + (\mathbf{d} + \mathbf{1})\mathbf{k})!}{\mathbf{k}!(3 + \mathbf{d} \cdot \mathbf{k})!} = \frac{(1 + (\mathbf{d} + \mathbf{1})\mathbf{k})!}{\mathbf{k}^i!(3 + \mathbf{d} \cdot \mathbf{k})!} = \frac{1}{d_i + 3} A(d_i + 3; \mathbf{k}^i)$$

The $(3 + \mathbf{d} \cdot \mathbf{k})$ term yields

$$\begin{aligned} (3 + \mathbf{d} \cdot \mathbf{k}) \frac{(1 + (\mathbf{d} + 1) \cdot \mathbf{k})!}{\mathbf{k}!(3 + \mathbf{d} \cdot \mathbf{k})!} &= \frac{(1 + (\mathbf{d} + 1) \cdot \mathbf{k})!}{\mathbf{k}!(2 + \mathbf{d} \cdot \mathbf{k})!} \\ &= \frac{1}{2} \frac{2}{2 + (\mathbf{d} + 1) \cdot \mathbf{k}} \binom{2 + (\mathbf{d} + 1) \cdot \mathbf{k}}{\mathbf{k}} \\ &= \frac{1}{2} A(2; \mathbf{k}) \end{aligned}$$

Thus we have

$$(4 + \mathbf{d} \cdot \mathbf{k}) \frac{1}{2 + (\mathbf{d} + 1) \cdot \mathbf{k}} \frac{(3 + (\mathbf{d} + 1) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!} = \frac{A(2; \mathbf{k})}{2} + \sum_{i=1}^m \frac{A(d_i + 3; \mathbf{k}^i)}{d_i + 3}$$

We are left with the term

$$\begin{aligned} -(4 + \mathbf{d} \cdot \mathbf{k}) \frac{1}{3 + (\mathbf{d} + 1) \cdot \mathbf{k}} \frac{(3 + (\mathbf{d} + 1) \cdot \mathbf{k})!}{\mathbf{k}!(4 + \mathbf{d} \cdot \mathbf{k})!} &= -\frac{1}{3} \frac{3}{3 + (\mathbf{d} + 1) \cdot \mathbf{k}} \binom{3 + (\mathbf{d} + 1) \cdot \mathbf{k}}{\mathbf{k}} \\ &= -\frac{1}{3} A(3; \mathbf{k}) \end{aligned}$$

Combining the above, and recalling the coefficient of $\frac{3}{2}$, and summing over $\mathbf{k} \in (\mathbb{Z}^{\geq 0})^m$ we have the total contribution of the $\frac{3}{2}(4 + \mathbf{d} \cdot \mathbf{k})$ -terms:

$$(A.7) \quad \frac{3}{2} \left(\frac{1}{2} z^2 - \frac{1}{3} z^3 + \sum_{i=1}^m \frac{s_i z^{d_i+3}}{d_i + 3} \right)$$

A.9.3. *Combining terms.* To conclude, we combine equations (A.6) and (A.7) to obtain

$$\begin{aligned} G(\mathbf{s}) &= \frac{3}{2} \left(\frac{1}{2} z^2 - \frac{1}{3} z^3 + \sum_{i=1}^m \frac{s_i z^{d_i+3}}{d_i + 3} \right) - \\ &\quad - \sum_{i=1}^m \frac{d_i(d_i + 4)}{2(d_i + 2)} \left(\frac{s_i z^{d_i+3}}{d_i + 3} - \frac{s_i z^{d_i+4}}{d_i + 4} + \sum_{j=1}^m \frac{s_i s_j z^{4+d_i+d_j}}{4 + d_i + d_j} \right). \end{aligned}$$

Collecting terms, we have

$$\begin{aligned} G(\mathbf{s}) &= \frac{3}{4} z^2 - \frac{1}{2} z^3 + \sum_{i=1}^m \frac{s_i z^{d_i+3}}{d_i + 3} \left(\frac{3}{2} - \frac{d_i(d_i + 4)}{2(d_i + 2)} \right) + \\ &\quad + \frac{1}{2} \left(\sum_{i=1}^m \frac{d_i s_i z^{d_i+4}}{d_i + 2} - \sum_{i,j=1}^m \frac{d_i(d_i + 4)}{d_i + 2} \frac{s_i s_j z^{4+d_i+d_j}}{4 + d_i + d_j} \right) \end{aligned}$$

Finally, using

$$\frac{3}{2} - \frac{d_i(d_i + 4)}{2(d_i + 2)} = -\frac{1}{2} \frac{(d_i + 3)(d_i - 2)}{d_i + 2},$$

we get

$$\frac{s_i z^{d_i+3}}{d_i + 3} \left(\frac{3}{2} - \frac{d_i(d_i + 4)}{2(d_i + 2)} \right) = -\frac{1}{2} \frac{d_i - 2}{d_i + 2} z^{d_i+3}.$$

Substituting this into our expression for G , we obtain as desired

$$G(\mathbf{s}) = \frac{3}{4}z^2 - \frac{1}{2}z^3 + \frac{1}{2} \left(\sum_{i=1}^m \frac{d_i s_i z^{d_i+4}}{d_i+2} - \sum_{i=1}^m \frac{(d_i-2)s_i z^{d_i+3}}{d_i+2} - \sum_{i,j=1}^m \frac{d_i(d_i+4)s_i s_j z^{4+d_i+d_j}}{(d_i+2)(4+d_i+d_j)} \right)$$

□

B. COUNTING PILLOWCASE COVERS

In this Appendix we describe the original approach to calculating the volume of the moduli space of Abelian and quadratic differentials suggested by H. Masur, M. Kontsevich, and the authors, and developed with success by A. Eskin and A. Okounkov, see [EO01, EO03]. This approach was also used in [Z00] and [EMS03]. The key idea is to translate the volume calculation into a counting problem for “integer points”, which geometrically correspond to *square-tiled surfaces* for the moduli spaces of Abelian differentials and to *pillowcase covers* for the moduli spaces of quadratic differentials.

In §B.1 we show why volume calculation is equivalent to counting the lattice points. In §B.2 we recall the definitions of the *pillowcase orbifold* and of the *pillowcase cover* and show that counting of lattice points is equivalent to the counting problem for pillowcase covers and prove Theorem 1.6.

B.1. Reduction of volume calculation to counting lattice points. The volume of a stratum $\mathcal{Q}_1(d_1, \dots, d_k)$ is defined by (4.5) as

$$\text{Vol } \mathcal{Q}_1(d_1, \dots, d_k) = \dim_{\mathbb{R}} \mathcal{Q}(d_1, \dots, d_k) \cdot \mu(C(\mathcal{Q}_1(d_1, \dots, d_k))),$$

where $\mu(C(\mathcal{Q}_1(d_1, \dots, d_k)))$ is the total volume of the “cone” $C(\mathcal{Q}_1(d_1, \dots, d_k)) \subset \mathcal{Q}(d_1, \dots, d_k)$ measured by means of the volume element $d\mu$ on $\mathcal{Q}(d_1, \dots, d_k)$ defined in §4.1. The total volume of the cone $C(\mathcal{Q}_1(d_1, \dots, d_k))$ is the limit of the appropriately normalized Riemann sums.

The volume element $d\mu$ is defined as a linear volume element in cohomological coordinates, normalized by certain specific lattice. Chose a positive ε such that $1/\varepsilon$ is integer, and consider a sublattice of the initial lattice of index $(1/\varepsilon)^{\dim_{\mathbb{R}} \mathcal{Q}(d_1, \dots, d_k)}$ partitioning every side of the initial lattice into $1/\varepsilon$ pieces. The corresponding Riemann sums count the number of points of the sublattices which get inside the cone. Thus, by definition of the measure μ we get

$$\mu(C(\mathcal{Q}_1(d_1, \dots, d_k))) = \lim_{\varepsilon \rightarrow 0} \varepsilon^{\dim_{\mathbb{R}} \mathcal{Q}(d_1, \dots, d_k)} \cdot (\text{Number of points of the } \varepsilon\text{-sublattice inside the cone } C(\mathcal{Q}_1(d_1, \dots, d_k))).$$

We assume that $1/\varepsilon$ is integer. Note that a flat surface S represents a point of the ε -lattice, if and only if the surface $(1/\varepsilon) \cdot S$ (in the sense of definition (4.1)) represents a point of the integer lattice. Denoting by $C(\mathcal{Q}_N(d_1, \dots, d_k))$ the set of flat surfaces in the stratum $\mathcal{Q}(d_1, \dots, d_k)$ of area at most $N/2$, and taking into consideration that

$$\text{area}((1/\varepsilon) \cdot S) = 1/\varepsilon^2 \cdot \text{area}(S)$$

we can rewrite the above relation as

$$(B.1) \quad \mu(C(\mathcal{Q}_1(d_1, \dots, d_k))) = \lim_{N \rightarrow +\infty} N^{-\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k)} \cdot (\text{Number of lattice points inside the cone } C(\mathcal{Q}_N(d_1, \dots, d_k))).$$

B.2. Lattice points, square-tiled surfaces, and pillowcase covers. Let $\Lambda \subset \mathbb{C}$ be a lattice, and let $\mathbb{T}^2 = \mathbb{C}/\Lambda$ be the associated torus. The quotient

$$\mathcal{P} := \mathbb{T}^2 / \pm$$

by the map $z \rightarrow -z$ is known as the *pillowcase orbifold*. It is a sphere with four $(\mathbb{Z}/2)$ -orbifold points (the corners of the pillowcase). The quadratic differential $(dz)^2$ on \mathbb{T}^2 descends to a quadratic differential on \mathcal{P} . Viewed as a quadratic differential on the Riemann sphere, $(dz)^2$ has simple poles at corner points. When the lattice Λ is the standard integer lattice $\mathbb{Z} \oplus i\mathbb{Z}$, the flat torus \mathbb{T}^2 is obtained by isometrically identifying the opposite sides of a unit square, and the pillowcase \mathcal{P} is obtained by isometrically identifying two squares with the side $1/2$ by the boundary, see Figure 12.

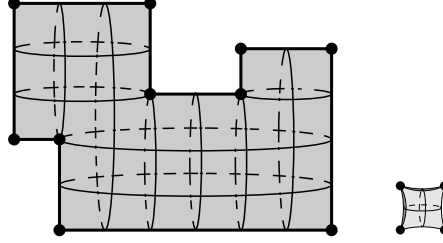


FIGURE 12. Pillowcase cover (on the left) over the pillowcase orbifold (on the right). A general pillowcase cover it is not necessarily glued from two identical polygons.

Consider a connected ramified cover $\hat{\mathcal{P}}$ over \mathcal{P} of degree N having ramification points only over the corners of the pillowcase. Clearly, $\hat{\mathcal{P}}$ is tiled by $2N$ squares of the size $(1/2) \times (1/2)$ in such way that the squares do not superpose and the vertices are glued to the vertices. Coloring the two squares of the pillowcase \mathcal{P} one in black and the other in white, we get a chessboard coloring of the square tiling of the the cover $\hat{\mathcal{P}}$: the white squares are always glued to the black ones and vice versa.

Lemma B.1. *Let S be a flat surface in the stratum $\mathcal{Q}(d_1, \dots, d_k)$. The following properties are equivalent:*

- (1) *The surface S represents a lattice point in $\mathcal{Q}(d_1, \dots, d_k)$;*
- (2) *S is a cover over \mathcal{P} ramified only over the corners of the pillow;*
- (3) *S is tiled by black and white $(1/2) \times (1/2)$ squares respecting the chessboard coloring.*

Proof. We have just proved that (2) implies (3). To prove that (1) implies (2) we define the following map from S to \mathcal{P} . Fix a zero or a pole P_0 on S . For any $P \in S$ consider a path $\gamma(P)$ joining P_0 to P having no self-intersections and having no zeroes or poles inside. The restriction of the quadratic differential q to such $\gamma(P)$

admits a well-defined square root $\omega = \pm\sqrt{q}$, which is a holomorphic form on the interior of γ . Define

$$P \mapsto \left(\int_{\gamma(P)} \omega \mod \mathbb{Z} \oplus i\mathbb{Z} \right) / \pm .$$

Of course, the path $\gamma(P)$ is not uniquely defined. However, since the flat surface S represents a lattice point (see the definition in §4.1), the difference of the integrals of ω over any two such paths $\gamma_1(P)$ and $\gamma_2(P)$ belongs to $\mathbb{Z} \oplus i\mathbb{Z}$, so taking the quotient over the integer lattice and over \pm we get a well-defined map. By definition of the pillowcase \mathcal{P} we have, $\mathcal{P} = (\mathbb{C} \mod \mathbb{Z} \oplus i\mathbb{Z}) / \pm$. Thus, we have defined a map $S \rightarrow \mathcal{P}$. It follows from the definition of the map, that it is a ramified cover, and that all regular points of the flat surface S are regular points of the cover. Thus, all ramification points are located over the corners of the pillowcase.

A similar consideration shows that (3) implies (1). \square

Let $\text{Sq}_N(d_1, \dots, d_k)$ be the number of surfaces in the stratum $\mathcal{Q}(d_1, \dots, d_k)$ tiled with at most N black and N white squares respecting the chessboard coloring. Lemma B.1 allows us to rewrite formula (B.1) as follows:

$$\mu(C(\mathcal{Q}_1(d_1, \dots, d_k))) = \lim_{N \rightarrow +\infty} N^{-\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k)} \cdot \text{Sq}_N(d_1, \dots, d_k).$$

Taking into consideration (4.5) we get

$$(B.2) \quad \text{Vol } \mathcal{Q}_1(d_1, \dots, d_k) = 2 \dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k) \cdot \lim_{N \rightarrow +\infty} N^{-\dim_{\mathbb{C}} \mathcal{Q}(d_1, \dots, d_k)} \cdot \text{Sq}_N(d_1, \dots, d_k).$$

Now everything is ready to prove Theorem 1.6.

Proof of Theorem 1.6. Consider a cover of $\pi : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$ having ramification data (η, ν) , as in §1.4. The condition 1.9 ensures that such a cover is realizable, and that the genus of the cover is really zero. As the base sphere chose the pillowcase \mathcal{P} endowed with the quadratic differential $q_0 = (dz)^2$. It has four simple poles at the corners of the pillow and no other singularities. Pulling back $(dz)^2$ via π gives a quadratic differential on the covering \mathbb{CP}^1 with zeros (and possibly simple poles) of multiplicities $\{\nu_i - 2\}$ and $\{2\eta_j - 2\}$ and with no other singularities. Thus, by construction the pillowcase cover $\hat{\mathcal{P}} := (\mathbb{CP}^1, \pi^*q_0)$ belongs to $\mathcal{Q}(\eta, \nu)$.

By [EO01] the leading term in the asymptotics for number of pillowcase covers is the same as if we consider the pillowcase covers ramified only over the corners of our pillow, in such way that all conical singularities of $\hat{\mathcal{P}}$ project to the same corner of the pillow, and all preimages of three other corners of the pillow \mathcal{P} are regular points of the metric. From now on we consider only such *square-tiled* pillowcase covers.

Since all preimages of three corners are regular points of the flat metric, they can be tiled with 2×2 squares, or, equivalently, each pillowcase cover under consideration decomposes into two consecutive covers

$$\pi : \hat{\mathcal{P}} \rightarrow \mathcal{P}_4 \rightarrow \mathcal{P},$$

where $\mathcal{P}_4 \rightarrow \mathcal{P}$ is a cover of order 4 of a pillowcase orbifold of size 2×2 over the standard pillowcase. Thus a pillowcase cover of degree $4d$ with ramification data (η, ν) uniquely defines a square-tiled pillowcase cover of degree d in $\mathcal{Q}(\eta, \nu)$.

Reciprocally, consider an arbitrary square-tiled surface S as in Lemma B.1 above in the stratum $\mathcal{Q}(\eta, \nu)$, and let d be the degree of the corresponding cover over \mathcal{P} . Subdividing each square into four; considering the underlying pillowcase as \mathcal{P}_4 and postcomposing the initial cover $S \rightarrow \mathcal{P}_4$ with the cover $\mathcal{P}_4 \rightarrow \mathcal{P}$ we get a pillowcase cover with singularity pattern (η, ν) . This implies that

$$\sum_{d=1}^N \text{Cov}_{4d}^0(\eta, \nu) \sim \text{Sq}_N(\nu_1 - 2, \nu_2 - 2, \dots, 2\eta_1 - 2, 2\eta_2 - 2, \dots).$$

Taking into consideration that

$$\dim_{\mathbb{C}} \mathcal{Q}(\nu_1 - 2, \nu_2 - 2, \dots, 2\eta_1 - 2, 2\eta_2 - 2, \dots) = \ell(\nu) + \ell(\eta) - 2$$

and applying (B.2) we complete the proof of Theorem 1.6. \square

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